Weinstein Conjecture and GW Invariants

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1 Introduction

The purpose of this paper is to provide a new and general method to find closed orbits for the characteristic foliation on a compact hypersurface of contact type by using Gromov-Witten invariants. The question on whether such closed orbits exist has been known as the Weinstein conjecture, proposed in [W]. As one of the applications of our method, we completely solve a stabilized version of this conjecture in this paper. To describe the conjecture, we need to introduce some basic notations first.

Let V be a connected, symplectic manifold with a symplectic form ω. A hypersurface S is said to be of contact type if there exists a vector field X defined on some neighborhood U of S such that (i) X is transversal to S and (ii) $L_X\omega = \omega$.

Now for any hypersurface S in a symplectic manifold V, there exists a 1-dimensional characteristic foliation ξ of S defined by:

$$\xi_x = \{v_x \,|\, v_x \in T_xS,\, \omega(v_x, u_x) = 0, for\ all\ u_x \in T_xS\}$$

for $x \in S$.

The Weinstein conjecture claims that if S is of contact type and compact, then S carries at least one closed orbit of ξ. The stabilized version of this conjecture claims the same conclusion as above under the assumption that S is contained in $(V \times \mathbf{C}^n, \omega \oplus \omega_0)$, the stabilization of V.

Before we state our result, we recall that given $A \in H_2(V, \mathbf{Q})$, the $(n+2)$-pointed GW invariant is a homomorphism

$$\Psi^V_{A,g,n+2} : H_*(\bar{\mathcal{M}}_{g,n+2}, \mathbf{Q}) \times H_*(V, \mathbf{Q})^{n+2} \longrightarrow \mathbf{Q},$$

(see [FO] and [LT]). Here for convenience, we use homology instead of cohomology as in [FO] and [LT]. We will omit the upscript V if no confusion arises.

Throughout this paper, we will assume that S separates V, i.e. there exist two submanifolds V_- and V_+ of V with common boundary S such that $V_- \cup V_+ = V, V_- \cap V_+ = S$. This can be achieved by imposing, for example, that $H^1(V, \mathbf{Z}_2) = 0$.

The main theorems of this paper are stated as follows.

Theorem 1.1 *If there exist $A \in H_2(V, \mathbf{Z})$ and $\alpha_+, \alpha_-, \in H_*(V, \mathbf{Q})$, such that*
(i) $supp(\alpha_+) \hookrightarrow \overset{\circ}{V}_-$, and $supp(\alpha_-) \hookrightarrow \overset{\circ}{V}_+$;
(ii) the GW-invariant $\Psi_{A,g,n+2}(C; \alpha_-, \alpha_+, \beta_1, \cdots, \beta_n) \neq 0$,
then S carries at least one closed orbit of ξ.

In particular, we have

Theorem 1.2 *Let S be as above. If there exist $A \in H_2(V, \mathbf{Z})$ such that the invariant $\Psi_{A,g,n+2}(\cdot; e, e, \cdots) \neq 0$, where e denotes the generator of $H_0(V, \mathbf{Z})$ represented by a point, then S carries at least one closed orbit of ξ.*

Among various potential applications of these two theorems, we only mention the following corollaries.

As a corollary to Theorem 1.1, we have completely solved the stabilized Weinstein conjecture in the following theorem.

Theorem 1.3 *The Weinstein conjecture holds for $(V \oplus \mathbf{C}^l, \omega \oplus \omega_0)$. That is, after V is stabilized by $\mathbf{C}^l$, the Weinstein conjecture holds.*

As a corollary to Theorem 1.2, we have

Theorem 1.4 *The Weinstein conjecture holds for $\prod_{i=1}^k \mathbf{C}P^{n_i}$ with the symplectic form $\omega = \oplus_{i=1}^k \omega_i$, where ω_i is the standard symplectic from of $\mathbf{C}P^{n_i}$. Moreover, the Weinstein conjecture holds for any rational algebraic manifolds (V, ω), provided there is a surjective morphism $\pi : V \to CP^n$ such that π is one to one over $V \setminus S$ for some subvariety S of V with $codim_{\mathbf{C}} \pi(S) \geq 2$. In particular,the Weinstein conjecture holds for any blow-ups of $\mathbf{C}P^n$ along its subvarieties.*

A special case of this theorem, where $V = \mathbf{C}P^n$, was proved by Hofer and Viterbo in [HV].

Closely related to this conjecture is the existence of closed orbits of some Hamiltonian function, which can be described as follows.

Let $\Psi : S \times (-\epsilon, \epsilon) \to U \hookrightarrow V$ be the flow of the vector field X. Since X is transversal to S, Ψ is a diffeomorphism from $S \times (-\epsilon, \epsilon)$ to some neighborhood W of S in V. Let $S_t = \Psi(S \times \{t\})$, $W_- = \cup_{t<0} S_t$ and $W_+ = \cup_{t>0} S_t$. Note that $S = S_0$. Then $W = S \cup W_+ \cup W_-$.

Because of our assumption that S separates V, we may assume further that

$$
(*)\begin{cases}
\text{there exist two submanifolds } V_- \text{ and } V_+ \text{ of } V\\
\text{with common boundary } S \text{ such that}\\
\text{(i) } V_- \cup V_+ = V, V_- \cap V_+ = S;\\
\text{(ii) } W_- \hookrightarrow V_-, W_+ \hookrightarrow V_+.
\end{cases}
$$

This condition implies that S can be realized as a zero set of some Hamiltonian function.

A particular defining Hamiltonian function $\tilde{H} = \tilde{H}_{S,X}$ of S can be defined as follows.

$$
\tilde{H}(x) = \begin{cases} \epsilon, & x \in V_+ \setminus W_+ \\ \phi(t) & x \in S_t \\ -\epsilon & x \in V_- \setminus W_-, \end{cases}
$$

where $\phi : [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ is a smooth function defined by

$$
\phi(t) = \begin{cases} t & -\epsilon + 2\delta < t < \epsilon - 2\delta \\ -\epsilon & t < -\epsilon + \delta \\ \epsilon & t > \epsilon - \delta, \end{cases}
$$

for some $0 < \delta << \epsilon$. The Hamiltonian vector field $X_{\tilde{H}}$ is defined by

$$
\omega(X_{\tilde{H}}, \cdot) = d\tilde{H}.
$$

Consider the Hamiltonian equation

$$
\frac{dx}{dt} = X_{\tilde{H}}(x(t)). \tag{1}
$$

Any non-trivial closed orbit x of (1) will lie on some level hypersurface $S_t = \tilde{H}^{-1}(t)$, $-\epsilon + \delta < t < \epsilon - \delta$. Now the condition $\mathcal{L}_X \omega = \omega$ implies that the characteristic foliation ξ_t on S_t is conjugate to $\xi = \xi_0$ on S under the flow Ψ. It follows that S will also carry a closed orbit of ξ given by $\Psi_t^{-1}(x)$.

Therefore, the Weinstein conjecture for those compact hypersurfaces of contact type satisfying (*) can be proved as long as the existence of some non-trivial closed orbits of (1) can be established.

The Weinstein conjecture was first proved for a convex or star-shaped hypersurface in $(\mathbf{R}^{2n}, \omega_0)$ by Weinstein and Rabinowitz in [W] and [R] respectively. In 1986, a substantial progress was made by Viterbo in [V]. He proved the conjecture for any compact hypersurface of contact type of $(\mathbf{R}^{2n}, \omega_0)$. A simpler proof of this was given by Hofer and Zehnder in [HZ]. We notice here that for any hypersurface of $\mathbf{R}^{2n}$, the above condition (*) always holds. Due to the work of Gromov and Floer, it is possible to generalize this result to hypersurfaces in certain general symplectic manifolds. In [FHV], Floer, Hofer and Viterbo proved the Weinstein conjecture for $M \times \mathbf{R}^{2n}$ with symplectic form $\omega \oplus \omega_0$

under the assumption that $\pi_2(M) = 0$. Note that any compact hypersurface S of $M \times \mathbf{R}^{2n}$ can be embedded into $M \times \{S^2_r\}^n$ for some large r, where S^2_r is the 2-dimensional sphere of radius r with the standard symplectic form ω_0 given by the area form. In [HV], Hofer and Viterbo proved the same statement under the weaker but rather technical assumption that $\min \omega(A) > \omega_0([S^2_r])$ for all effective classes $A \in H_2(M)$. Here a second homology class A is said to be effective if there exists an ω-compatible almost complex structure J and a non-trivial J-holomorphic sphere $f : S^2 \to V$ such that $[f] = A$. We note that in [FHV] and [HV], the condition (*) was never stated explicitly, although such a restriction seems to be necessary for their method of using Hamiltonian functions as we remarked above. We may view the main results obtained in [FHV] and [HV] as a stabilized version of the Weinstein conjecture. In this aspect, as we mentioned before, we are able to solve such a stabilized Weinstein conjecture completely without any restriction on (V, ω). (See Theorem 1.3 above.) For three dimensional contact manifolds, many deep results have been proved on the Weinstein conjecture and related problems by Eliashberg, Hofer, Zehnder and others (c.f. [EH], [HWZ]). For example, Hofer solved the Weinstein conjecture for overtwisted contact 3-manifolds.

The main focus of this paper, however, is not only to prove the stabilized Weinstein conjecture, but to establish the full relationship between the existence of J-holomorphic curves of any genus and the existence of non-trivial closed orbits of $\widetilde{H}$. Such a relationship obtained by using Gromov-Witten invariants of any genus did not appear in previous literature even for the semi-positive case.

The general idea of proving the existence of closed orbits for the Hamiltonian equation (1) by using genus zero J-holomorphic curves or perturbed J-holomorphic curves was already realized by Floer, Hofer and Viterbo. One quantitative form of such an idea was developed in [FHV] and [HV] as their theory of d-index. The desired existence results were then obtained by exploiting the deformation invariance of the d-index. However, the results obtained by this theory are quite limited. It may be partly because the well-known difficulty of the transversality of multiple covered J-holomorphic spheres of negative first Chern class and partly because the pathological nature of Hamiltonian function used in d-index. In fact, most of the results obtained in [HV] were not proved even for semi-positive symplectic manifolds before.

The recent progress on the Floer homology theory and GW-invariants (cf. [FO], [LT], [LiuT1]) enables us to overcome the difficulty of transversality. Furthermore, in this paper, we will describe the full relationship of the existence of J-holomorphic curves of any genus with the existence of closed orbits of the Hamiltonian equation (1) in its general form. We believe that this new finding will throw light on solving the Weinstein conjecture completely. Hopefully, this new finding also gives clues to understanding the mystery of Gromov-Witten invariants on symplectic manifolds.

To prove Theorem 1.1, we choose an ω-compatible almost complex structure

J and consider a family of Hamiltonian functions $\widetilde{H}_\lambda = \lambda \cdot \widetilde{H}$, $\lambda \in [0, \infty)$. Let $A \in H_2(V)$ and α_+, α_-, $\beta_j \in H_*(V)$, $j = 1, \cdots, n$. As before, we assume that $\alpha_- \in V_+$ and $\alpha_+ \in V_-$. The key step now is to obtain a Morse function H which is a small perturbation of $\tilde{H}$ such that $H_\lambda = \lambda \cdot H$ has no non-trivial closed orbits in $V \setminus W$ and has the same closed orbits as $\tilde{H}_\lambda$ has in W, if $0 < \lambda < 1 + \omega(A)/2\epsilon$. The reason for choosing the quantity $1 + \omega(A)/2\epsilon$ will be explained in Section 7 of this paper (see [HV] also). By the usual Morse theory, α_+ (α_-) can be represented by a linear combination of some critical points of H, denoted it by c_+ (c_-), together with the associated descending (ascending) manifold $M(c_+)$ ($M(c_-)$) in V_- (V_+).

We will define a perturbed GW-invariant $\Phi_{A, J_\lambda, H_\lambda, g}(c_-, c_+, \beta_1, \cdots, \beta_n)$ in Section 7 and Section 8. In genus zero case, it counts algebraically the ν-perturbed (J, H_λ)-maps

$$u : (\mathbf{R}^1 \times S^1; x_1, \cdots, x_n) \to (V; \beta_1, \cdots, \beta_n),$$

satisfying the conditions
(i) $\bar{\partial}_{J, F_\lambda, \nu} u = 0$, (ii) $\lim_{s \to +\infty} u(s, t) = c_+$, $\lim_{s \to -\infty} u(s, t) = c_-$, (iii) $[u] = A$ (see Section 8 for higher genus case).

Theorem 1.5 *When λ is small enough,*

$$\Phi_{A, J_\lambda, H_\lambda, g}(c_-, c_+, \beta_1, \cdots, \beta_n) = \Psi_{A, g}(\alpha_-, \alpha_+, \beta_1, \cdots, \beta_n).$$

This theorem were claimed in [PSS] and [RT] for semi-positive case. The third different method were described in [L2]. However, these methods are not sufficient for general symplectic manifolds. Using the techniques devoloped in [LiuT1], we will prove this theorem in [LiuT3].

The following theorem is the main technique part of this paper.

Theorem 1.6 *If H has no non-trivial closed orbits, then the perturbed GW-invariant $\Phi_{A, J_\lambda, H_\lambda, g}(c_-, c_+, \beta_1, \cdots, \beta_n)$ is well-defined, independent of the choice of $\lambda \in (0, 1 + \omega(A)/2\epsilon)$. Moreover, $\Phi_{A, J_\lambda, H_\lambda, g}(c_-, c+, \beta_1, \cdots, \beta_n) = 0$ when $\lambda > \omega(A)/2\epsilon + \frac{1}{2}$.*

The proof of this theorem requires a T^{N_P}-equivariant version of the new technique developed in [LT] and [LiuT1]. The simplest case of such a theory , the S^1-equivariant case, was already used in our computation of Floer homology in [LiuT1].

Now Theorem 1.1 follows from Theorem 1.5 and Theorem 1.6 easily.

The main body of this paper (from Section 3 to Section 7) is devoted to establish a Morse theoretic version of GW invariants of genus zero case for general symplectic manifolds under the assumption that H has no non-trivial closed orbits. We then prove Theorem 1.1 for genus zero case in Section 7. In the last section, we generalize the theory of genus zero case to higher genus case and prove Theorem 1.1 for any genus.

We note that if the symplectic manifold V is semi-positive, our theory in this paper can be developed in a much simpler manner. This includes the case where dimension of V is four or six.

This paper is the detailed version of our announcement [LiuT2]. During the preparation of this paper, we learned that W. Chen proved some relevant results for the 4-dimensional case in [C] by a different method.

From now on until the end of Section 7, we will only deal with the case of genus zero. The discussions for higher genus cases are identical and will be outlined in last section.

2 Compactness

In this section, we will set up our main assumption, which will be used throughout the rest of this paper. We then explore the two simple consequences of the assumption, the existence of the Morse function H mentioned in Section 1 and the compactness of the moduli spaces of cuspidal (J, H)-maps.

- **Main Assumption (I):** $\tilde{H}$ has no non-trivial closed orbits.

The first consequence of this assumption is the following lemma,

Lemma 2.1 *There exists a Morse function H such that (i) H has same level sets as $\tilde{H}$ has in W;*

(ii) H is C^0-close to $\tilde{H}$ so that for any critical points c_- in V_- and c_+ in V_+,

$$0 < \frac{\omega(A)}{H(c_-) - H_(c_+)} < \lambda_0,$$

where $\lambda_0 = \frac{1}{2} + \frac{\omega(A)}{2\epsilon}$ and A is an effective second homology class in the sense that it can be represented by some J-holomorphic sphere;

(iii) for any $0 < \lambda < \lambda_0 + \frac{1}{2}$, $H_\lambda = \lambda \cdot H$ has no non-trivial closed orbits of period one.

Proof:

Choose $r > 0$ such that

$$0 < \frac{\omega(A)}{2\epsilon - 2r - 4\delta} < \lambda_0.$$

Here δ is the same as the one appeared in the definition of $\tilde{H}$. As before we assume that $\delta << \epsilon$. Recall that $\tilde{H}(x) = t$ if $x \in S_t$, $-\epsilon + 2\delta < t < \epsilon - 2\delta$. Let V_s be the manifold $V_+ \setminus S \times [0, s]$ with boundary S_s. Set $\tilde{H}_+ = \tilde{H}|_{V_{\epsilon-2\delta}}$. Then $\nabla \tilde{H}_+ \neq 0$ along the boundary of $V_{\epsilon-2\delta}$.

It is well-known that there is a C^2-small C^∞-function $\tilde{G}_+ : V_{\epsilon-2\delta} \to \mathbf{R}$ such that $\tilde{F}_+ = \tilde{H}_+ + \tilde{G}_+$ is a Morse function on $V_{\epsilon-2\delta}$. Since $\tilde{H}_+$ is regular along $S_{\epsilon-2\delta}$, we may arrange that $\tilde{G}_+$ vanishes near $S_{\epsilon-2\delta}$. Now decompose $\tilde{F}_+$

as: $\tilde{F}_+ = \epsilon - 2\delta + \bar{F}_+$ and define $\tilde{F}^\lambda_+ = \epsilon - 2\delta + \lambda\bar{F}_+$. Let $\bar{F}^\lambda_+ = \lambda\bar{F}_+$. Then $\nabla\tilde{F}^\lambda_+ = \nabla\bar{F}^\lambda_+$. Hence $\tilde{F}^\lambda_+$ has non-trivial closed orbits if and only if $\bar{F}^\lambda_+$ has. Now $\|\bar{F}^\lambda_+\|_{C^2} = \lambda\|\bar{F}_+\|_{C^2}$, which implies that the C^2-norm of $\bar{F}^\lambda_+$ is small when λ is small enough. Therefore, there exists a $\lambda_1 > 0$ such that $\overline{F}^\lambda_+$ (hence $\tilde{F}^\lambda_+$) has no non-trivial closed orbits of period 1 for $0 < \lambda < \lambda_1$.

Similarly we can also define $\tilde{F}_-, \tilde{F}^\lambda_-$, etc on $V_{-\epsilon+2\delta}$.

Fix a $\lambda > 0$ satisfying the following two conditions:
(a) $\lambda(\lambda_0 + 1/2) < \lambda_1$;
(b) for any critical point c_+ of $\overline{F}_+$ and c_- of $\overline{F}_-$,

$$\lambda|\overline{F}_+(c_+) - \overline{F}_-(c_-)| < 2r.$$

Define $H_+ = \tilde{F}^\lambda_+$ and $H_- = \tilde{F}^\lambda_-$. We will extend $H_+ \cup H_-$ to V to get an H with the same level set as $\tilde{H}$ has in the "middle part" $S \times [-\epsilon + 2\delta, \epsilon - 2\delta]$. If this is done, then it follows from (a) and (b) that H has the required properties of the lemma.

We define H on $S \times (-\epsilon + 2\delta, \epsilon - 2\delta)$ to extend $H_+ \cup H_-$ as follows,

$$H(x) = \begin{cases} \tilde{H}(x), & x \in S \times (-\epsilon + 3\delta, \epsilon - 3\delta) \\ \psi(t(x)), & x \in S \times \{(-\epsilon + 2\delta, -\epsilon + 3\delta) \cup (\epsilon - 3\delta, \epsilon - 2\delta)\}. \end{cases}$$

Here $t(x)$ is the t-coordinate of x and $\psi : (-\epsilon+2\delta, -\epsilon+3\delta) \cup (\epsilon-3\delta, \epsilon-2\delta) \to \mathbf{R}$ is an increasing C^∞- function defined by requiring that
(i) on $(\epsilon-3\delta, \epsilon-2\delta)$, ψ connects smoothly the two functions $\psi_1(t) = t, t \leq \epsilon-3\delta$, and $\psi_2(t) = (\epsilon - 2\delta) + \lambda(t - (\epsilon - 2\delta)), t \geq \epsilon - 2\delta$;
(ii) ψ does similar thing on $(-\epsilon + 2\delta, -\epsilon + 3\delta)$.

Clearly H so defined has the same level sets as $\tilde{H}$ has in $S \times (-\epsilon+2\delta, \epsilon-2\delta)$. □

Our assumption now becomes

- **Main Assumption (II)**:
  $H_\lambda = \lambda \cdot H$ has no nontrivial closed orbits of period one, for $0 \leq \lambda \leq \lambda_0 + \frac{1}{2}$.

Later on we will make some C^∞-small generic perturbation of H. Since the perturbation can be made arbitrarily small, we will assume that the main assumption (II) also holds for those perturbed H.

We now state the consequence of the assumption on the compactness of the moduli space of cuspidal (J_λ, H_λ)-maps, where $0 < \lambda < \lambda_0 + 1/2$.

Lemma 2.2 *Fix any two critical points c_- and c_+ of H_λ, let $\{f_i\}$ be a sequence of $(J_{\lambda_i}, H_{\lambda_i})$-maps of class A connecting c_- and c_+, with $\lambda_i \in [\epsilon, \lambda_0 + 1/2]$ for some small $\epsilon > 0$. After reparametrization of the domain of f_i and taking subsequence, we have that $\{f_i\}$ weakly C^∞-converges to a cuspidal $(J_{\lambda_\infty}, H_{\lambda_\infty})$-map f_∞ of same class A connecting c_- and c_+.*

The domain Σ of a cuspidal (J_λ, H_λ)-map f is a union $\Sigma = \cup_{i=1}^{N_P} P_i \cup_{j=1}^{N_B} B_j$ of its principal components P_i and bubble components B_j. Each $P_i \cong \mathbf{R}^1 \times S^1$ and the collection $\{P_i\}$ form a chain. Each bubble component $B_j \cong S^2$ is attached to some P_i or some other B_k at some of its singular points. All components of Σ form a tree.

Definition 2.1 *A continuous map $f : \Sigma \to V$ is said to be cuspidal (J_λ, H_λ)-map of class A connecting c_- and c_+, if there exist $N_P + 1$ critical points $c_1, \cdots, c_{N_P+1}$ of H_λ, with $c_1 = c_-$, $c_{N_P+1} = c_+$ such that*
(i) $\bar{\partial}_{J_\lambda, H_\lambda} f_i^P = 0, \lim_{s\to-\infty} f_i^P(s,\theta) = c_i, \lim_{s\to\infty} f_i^P(s,\theta) = c_{i+1}$, $i = 1, \cdots, N_P$.
(ii) $\bar{\partial}_{J_\lambda} f_j^B = 0$.
(iii) $\sum_i [f_i^P] + \sum_j [f_j^B] = A$.

An analogy of this lemma, in which the Morse function H_λ is replaced by some generic time-dependent Hamiltonian function is proved in [F] section 3. The proof there can be easily adapted to our case as long as we can make sure that (i) H_λ has no non-trivial periodical orbits of period one for $0 < \lambda < \lambda_0 + 1/2$; (ii) all critical points c of H_λ, when considered as a trivial periodical orbit of the time-independent Hamiltonian function H_λ is non-degenerate in the sense of Floer homology. Now (i) follows from our main assumption and (ii) can be achieved by a small C^∞-perturbation of H. Note that (ii) implies that any (J, H)-map convergent to c along its ends will converge to c exponentially.

3 Moduli Space of Stable Maps

In this section, we will define the various moduli spaces of stable maps needed to define the Morse theoretical version of GW-invariant.

3.1 Stable Curves

Stable curves will appear as the domains of stable maps, which are to be defined below. From this Section up to Section 7 we will only consider semi-stable (connected) curves of genus zero. Geometrically such a curve Σ is a union of its components $\Sigma_l \cong S^2$ with only double points as its singularities, and its components form a tree ($H_1(\Sigma) = 0$).

We now define semi-stable $\mathcal{F}$-curves and $\mathcal{G}$-curves:

Definition 3.1 *An n-pointed semi-stable $\mathcal{F}$-curves (Σ, l, x) is a semi-stable curve Σ with n (ordered) marked points $x = \{x_1, \cdots, x_n\}$ in Σ away from its singular points such that the components of Σ can be divided into principal components $P_i, i = 1, \cdots, N_P$, and bubble components $B_j, j = 1, \cdots, N_B$. The principal components form a chain in such a way that each P_i has two distinguished points z_i and z_{i+1}, $i = 1, \cdots, N_P$ such that P_i and P_{i+1} join together at z_{i+1}. l is the collection of marked lines l_i on P_i connecting its "ends" z_i and z_{i+1}.*

Using the marked line l_i , we may identify $(P_i \setminus \{z_i, z_{i+1}\}; l_i)$ with $(\mathbf{R} \times S^1; \{\theta = 0\})$.

An n-pointed semi-stable $\mathcal{G}$-curve (Σ, x) can be obtained from the corresponding $\mathcal{F}$-curve by simply forgetting all marked lines l_i's.

Two semi-stable $\mathcal{F}$-curves $(\Sigma^1; l^1, x^1)$ and $(\Sigma^2; l^2, x^2)$ are said to be equivalent if there is a homomorphism $\phi : \Sigma_1 \to \Sigma_2$ which preserves marked points and lines such that the restriction of ϕ to any component of Σ_1 is a biholomorphic map. We will use $\langle \Sigma, l, x\rangle$ to denote the resulting equivalence class of (Σ, l, x). Similarly we can define equivalence class for semi-stable $\mathcal{G}$-curves by simply forgetting those marked lines in the definition of the equivalence of $\mathcal{F}$-curves, and we will use $\langle \Sigma, x\rangle$ to denote the equivalence class of a semi-stable $\mathcal{G}$-curve (Σ, x).

Definition 3.2

$$\mathcal{FM}_{0,n} = \{\langle \Sigma, l, x\rangle | (\Sigma, l, x) \textit{ is a semi-stable } \mathcal{F}\textit{-curve}\},$$

$$\mathcal{GM}_{0,n} = \{\langle \Sigma, x\rangle | (\Sigma, x) \textit{ is a semi-stable } \mathcal{G}\textit{-curve}\}.$$

There is an obvious forgetting map:

$$\mathcal{FM}_{0,n} \to \mathcal{GM}_{0,n}$$

sending $\langle \Sigma, l, x\rangle$ to $\langle \Sigma, x\rangle$.

From now on, for simplicity, we will call a semi-stable $\mathcal{F}$-curve or a semi-stable $\mathcal{G}$-curve an $\mathcal{F}$-curve or $\mathcal{G}$-curve respectively.

Given an $\mathcal{F}$-curve (Σ, l, x) or a $\mathcal{G}$-curve (Σ, x), there is an obvious way to add minimal number of markings y_i to an unstable principal component P_i and $y_j^k, 1 \le k \le 2$, to an unstable bubble component B_j to stabilize Σ. We will use y to denote the set of the added markings and $(\Sigma, l, x; y)$ and $(\Sigma, x; y)$ to denote the resulting stabilized $\mathcal{F}$-curve and $\mathcal{G}$-curve and call them stable $\mathcal{F}$-curve and stable $\mathcal{G}$-curve respectively. Here stability means that each of its components contains at least three singular points or marked points in x or y. There is an obvious forgetting map here from the set of stable $\mathcal{F}$-curves or $\mathcal{G}$-curves to the set of semi-stable ones, sending $(\Sigma, l, x; y)$ to (Σ, l, x) or $(\Sigma, x; y)$ to (Σ, x) respectively.

From now on we will use various simplified notations, depending on the context, to denote above curves. For instance we may write (Σ, y) for $(\Sigma, l, x; y)$, if no confusion arises.

• **Local Deformation of** $(\Sigma, l, x; y)$

Given a stable $\mathcal{F}$-curve $(\Sigma; l, x; y)$, with double points

$$d_m^{P_i} \in P_i, m = 1, \cdots, M^{P_i} \quad \text{and} \quad d_l^{B_j} \in B_j, l = 1, \cdots, L^{B_j},$$

let $\alpha_m^{P_i}$ and $\alpha_l^{B_j}$ be the complex coordinates of the corresponding double points of $d_m'^{P_i}$ and $d_l'^{B_j}$ of a nearby curve Σ' of same topological type. Let $\alpha =$

$\{\alpha_m^{P_i}, \cdots, \alpha_l^{B_j}\}$, $m \leq M^{P_i} - 3 + r^{P_i}$, and $l \leq L^{B_j} - 3 + r^{B_j}$, where $r^{P_i}$and r^{B_j} are the number of elements in x and y in P_i and B_j respectively. Let θ be the collection of all angular coordinates θ_i of the third from last double or marked point of the principal components P_i'. Now $u = (\alpha, \theta)$ gives rise to the universal local coordinate of nearby stable $\mathcal{F}$-curve . We will use Σ_u to denote the nearby curve with coordinate u.

For each double point in $\Sigma' = \Sigma_u$, say, $d_1' \in P_1'$ and $d_2' \in B_2'$ with $d_1' = d_2'$ in Σ', we associate a complex gluing parameter $t_1 = t_2 \in D_\delta = \{z \,|\, |z| < \delta\}$. The corresponding gluing here is the following: cut off the two discs of radius $|t_1| = |t_2|$ of P_1' and B_2' centered at d_1' and d_2' respectively and glue them back along the boundary circles through a rotation of arg t_i. Let t be the collection of all such gluing parameters. Similarly for each $z_i, i = 2, \cdots, N_P - 1$, we associate a gluing parameter $\tau_i \in I_\delta = \{r \,|\, r \in \mathbf{R}^+, r < \delta\}$. There is also a similar but simpler gluing process for each τ_i. Let $\tau = (\tau_i)$ and $v = (t, \tau)$. Then $(\Sigma_{(u,v)} = (\Sigma_{(\alpha,\theta,t,\tau)}, l))$ is the local "universal " deformation of (Σ, l) as an $\mathcal{F}$-curve .

The "universal" deformation for $\mathcal{G}$-curve can be defined similarly. Since in this case there is no such marked lines l appearing, there is no such parameter θ and associate to each z_i is a complex parameter $\tilde{\tau}_i = (\tau_i, \theta_i)$ instead of τ_i. Let $\tilde{t}$ be the collection of all complex gluing parameters associated with double points and "ends" of Σ, the $\Sigma_{(\alpha,\tilde{t})}$ is the "universal" deformation of Σ as a $\mathcal{G}$-curve .

- **Fixed Markings**

Recall that in order to define GW-invariants, one needs to specify a cycle C in $H_*(\bar{\mathcal{M}}_{g,n+2}$. Chosing such a cycle will impose restrictions to the possible domains in the bubbling process of the Gromov-Floer compactification of stable maps. For simplicity, we only describe in detail the case $C = \{pt\}$. The general case can be treated similarly.

Since the main issue here only involves how to fix marked points, we can treat both $\mathcal{F}$-curve s and $\mathcal{G}$-curve s equally We only formulate the "fixed marking" process for $\mathcal{G}$-curve s.

Let $(S^2; -\infty, +\infty; \tilde{x}_1, \cdots, \tilde{x}_n)$ be a fixed a model, where $-\infty$ and $+\infty$ are the two "ends" if we identify $S^2 \setminus \{-\infty, +\infty\}$ with $\mathbf{R}^1 \times S^1$. We want to define the notion of a semi-stable curve with "fixed" marked points. markings, $x_1, \cdots, x_n$ (modeled on $(S^2; \tilde{x})$) if (i) there exists a principal component P_i and n many of its double points or marked points, $d_1, \cdots, d_n$ such that $(P_i; z_i, z_{i+1}; d_1, \cdots, d_n) \cong (S^2; -\infty, +\infty; \tilde{x}_1, \cdots, \tilde{x}_n)$; (ii) each marked point x_i of (Σ, x) lies on the branch $B(d_i)$ consisting of all bubble components with " root" d_i, if $x_i \neq d_i$.

Let (Σ, x, y) be the minimal stabilization of (Σ, x), the next lemma explains why the above two conditions are the desired ones.

Lemma 3.1 *There is a gluing procedure such that for any gluing parameter $\tilde{t}$ with non zero components,*

$$(\Sigma_{(\alpha,\tilde{t})}; z_1, z_{N_P+1}, x_1, \cdots, x_n) \cong (S^2; -\infty, +\infty; \tilde{x}_1, \cdots, \tilde{x}_n),$$

after forgetting those markings y of $(\Sigma_{(\alpha,\tilde{t})}, x, y)$.

Note here the parameter α is subject to the restriction imposed by the "fixed" marking condition.

Proof:

Away from those components $B_{i,j}$ in $B(d_i)$, the gluing procedure is the same as before. Let $x_i \in B_{i,l}$. Because of the tree structure of the components of Σ, there is a unique chain of bubble components $B_{i,j}, j = 1, \cdots, l$ of $B(d_i)$ with each $B_{i,j}$ having two particular double points d^j and d^{j+1}, connecting d_i and x_i. Here $d_i = d^1$ and $x_i = d^{l+1}$. Now there is a unique identification of each $B_{i,j} - \{d^j, d^{j+1}\} \cong \mathbf{R}^1 \times S^1$ up to translations and rotations of $\mathbf{R}^1 \times S^1$. Use the cylindrical coordinate here (or the corresponding polar coordinate) to do the gluing associated with the double points d^j. It is easy to see that after forgetting markings other than x, $\Sigma_{\alpha,\tilde{t}}$ has the desired property.

$\square$

3.2 Stable Maps

• Given a homology class $A \in H_2(V, \mathbf{Z})$, a stable (J, H)-map f from an $\mathcal{F}$-curve (Σ, l) to V of class A connecting critical point c_- and c_+ of H is a map defined on $(\Sigma, l) \setminus \bigcup_{i=1}^{N_P+1}\{z_i\}$ such that:

(i) on each principal component P_i,

$$\frac{\partial f_i^P}{\partial s} + J(f_i^P)\frac{\partial f_i^P}{\partial \theta} - \nabla H(f_i^P) = 0,$$

where $(s, \theta) \in \mathbf{R}^1 \times S^1$ is the cylindrical coordinate of P_i and $f_i^P = f|_{P_i - \{z_i, z_{i+1}\}}$;

(ii) there exist $c_i, c = 1, \cdots, N_P + 1$ with $c_1 = c_-, c_{N_P+1} = c_+$ such that

$$\lim_{s \to -\infty} f_i^P(s, \theta) = c_i \text{ and } \lim_{s \to +\infty} f_i^P(s, \theta) = c_{i+1};$$

(iii) on each bubble component B_j, $\bar{\partial}_J f_j^B = 0$;

(iv) $\sum_i [f_i^P] + \sum_j [f_j^B] = A$;

(v) each constant component is stable in the sense that it has at least three double or marked points.

Two such maps f_1 and f_2 with $\mathcal{F}$-curves as their domains are said to be equivalent if there is an identification

$$\phi : (\Sigma^1, l^1, x^1) \to (\Sigma^2, l^2, x^2)$$

such that $f_2 = f_1 \circ \phi$. Similarly, we can define equivalent relation for stable (J, H)-maps with $\mathcal{G}$-curve s as domains. We will use $\langle f \rangle$ to denote the resulting equivalence class of f.

We also need the notion of stable L_k^p-maps, which can be defined by simply requiring that f is a L_k^p-map, $k-\frac{2}{p}>1$ satisfying requirements (ii) , (iv) and (v).

Each stable map f determines an intersection pattern D_f which encodes the following information:
(i) the topological type of the domain $\Sigma=\Sigma_f$;
(ii) the homology classes $[f_i^P],[f_j^B]\in H_2(V,\mathbf{Z})$;
(iii) the critical points $c_i, i=1,\cdots N_P+1$.

Note that the topological type of Σ is determined by its intersection pattern $I=I_\Sigma$, which can be thought as a pairwise correspondence of the double points of Σ lifted to the smooth resolution of Σ.

Given a stable map f, we define its energy

$$E(f)=\sum_i E(f_i^P)+\sum_j\int_{S^2}(f_j^B)^*\omega,$$

where $E(f_i^P)=\int\int_{\mathbf{R}^1\times S^1}|\frac{\partial f_i^P}{\partial s}|^2$.

Note that if f is a (J,H)-map of class A connecting c_- and c_+, then

$$E(f)=\omega(A)+f(c_+)-f(c_-).$$

Lemma 3.2 *For a generic choice of (J,H), there exists a $\delta=\delta(J,H)>0$, such that for any non-constant stable (J,H)-map f, $E(f)>\delta$.*

Proof:

Assume that there exists a sequence of (J,H)-maps $\{f_i\}$ of class A such that $\lim E(f_i)=0$. By choosing a suitable subsequence we may assume that each f_i has only one principal component and connects two fixed critical points c_- and c_+. It follows from Gromov-Floer compactness theorem for cuspidal maps that a subsequence of $\{f_i\}$, still denoted by $\{f_i\}$, is C^0-convergent to a constant map. Hence $c_-=c_+$, $[f_i]=0$, for large i. If f_i is not a constant, there exists a unique simple $(J,\frac{1}{m}H)$-map $\tilde{f}_i$ such that $f_i=\tilde{f}_i\circ\pi_m$, where $\pi_m:\mathbf{R}^1\times S^1\to\mathbf{R}^1\times S^1$ is given by $\pi_m(s,\theta)=(ms,m\theta)$. Here $\tilde{f}_i$ being simple means that it can not be factorized through further for any $m>1$.

Consider the moduli space

$$\begin{aligned}&\mathcal{M}^0(c_-,c_+;J,H,A)\\&=\{g\,|\,g:\mathbf{R}^1\times S^1\to V\text{ is a }(J,H)\text{-map},[g]=A,g\text{ is simple}\}.\end{aligned}$$

Then for a generic choice of (J,H),

$$\dim\mathcal{M}^0(c_-,c_+;J,\frac{1}{m}H,\frac{1}{m}A)=Ind(c_+)-Ind(c_-)+2c_1(A)/m,$$

which is zero in the case that $c_- = c_+$ and $A = 0$. Clearly

$$\tilde{f}_i \in \mathcal{M}^0(c_-, c_+; J, \frac{1}{m}H, 0).$$

Since $\tilde{f}_i$ is not a constant, it follows from [FHS] that for a generic choice of (J, H), $\tilde{f}_i$ has a two dimensional symmetries, which implies that

$$\dim \mathcal{M}^0(c_-, c_+; J, \frac{1}{m}H, 0) \geq 2.$$

This is a contradiction.

□

An intersection pattern D is said to be effective if $D = D_f$ with f being a stable (J, H)-map. Let $e > 0$ and define

$$\mathcal{D}^e = \{D \mid D \text{ is effective, } E(D) \leq e\},$$

where the energy $E(D) = E(D_f) = E(f)$.

Lemma 3.3 *$\mathcal{D}^e$ is finite for any $e > 0$.*

Proof:

It follows form Gromov-Floer compactness theorem for cuspidal maps that there are at most finitely many possible homology classes which can be represented by some (J, H)-map f with $D_f \in \mathcal{D}^e$. Therefore it is sufficient to prove that there are only finitely many possible topological types of Σ_f for such f. To this end, we observe that f has at most $[\frac{e}{\delta}] + 1$ non-trivial component. This implies that the stabilized curve (Σ, y) obtained by adding minimal number of markings to Σ has at most $2([\frac{e}{\delta}] + 1) + n$ markings. This in turn bounds the number of double points, and hence bounds the number of components of Σ_f.

□

There is a partial order relation in $\mathcal{D}^e$ defined as follows : $D_1 = D_{f_1} \leq D_2 = D_{f_2}$ if (i) Σ_{f_2} can be obtained from Σ_{f_1} topologically by the gluing construction described in Section 2.1; (ii) the homological classes represented by the components of f_1 and f_2 are compatible with the gluing construction. (See the next subsection for the definition of the gluing of stable maps.)

3.3 Moduli Spaces of Stable Maps

Now we can define various moduli spaces of stable maps.

Let $\mathcal{FM}(c_-, c_+; J, H, A)$ be the moduli space of equivalence classes of stable (J, H)-maps of class A connecting c_- and c_+ with $\mathcal{F}$-curves as domains.

Similarly we can define the moduli space $\mathcal{GM}(c_-, c_+; J, H, A)$ of the equivalence classes of stable (J, H)-maps of class A connecting c_- and c_+ with $\mathcal{G}$-curves as domains.

Let $\mathcal{FB}^e(c_-, c_+; A)$ be the moduli space of equivalence classes of stable L_k^p-maps of class A connecting c_- and c_+ with $\mathcal{F}$-curves as domains, the energy of whose elements is less than e.

Since the energy $E(f)$ is bounded for any element in $\mathcal{FM}(c_-, c_+; J, H, A)$, $\mathcal{FM}(c_-, c_+; J, H, A) \subset \mathcal{FB}^e(c_-, c_+; A)$ when e is large enough. We will choose such an e once for all and omit the superscript e for the moduli space of L_k^p-maps.

Similarly we define $\mathcal{GB}(c_-, c_+; A)$.

We can also restrict to some particular intersection pattern $D \in \mathcal{D}^e$ and define the corresponding moduli spaces. We denote them by

$$\mathcal{FM}^D(c_-, c_+; J, H, A) \quad \text{and} \quad \mathcal{GM}^D(c_-, c_+; J, H, A) \quad \text{etc.}$$

From now on, we will omit c_- and c_+ in our notations of above moduli spaces when no confusion arises.

• **Weak topology on $\mathcal{FM}(J, H, A)$ and $\mathcal{GM}(J, H, A)$**

There are two different but equivalent topology on the moduli spaces of stable (J, H)-maps, the weak C^∞-topology and strong L_k^p- topology.

We start with defining the weak C^∞-topology. We will only deal with $\mathcal{FM}$ and leave the corresponding statements for $\mathcal{GM}$ to readers.

• Definition of Weakly Convergence

Given a sequence $\{\langle f_i \rangle\}_{i=1}^\infty$ of equivalence classes of stable (J, H)-maps with $\mathcal{F}$-curves as domains, we say that $\{\langle f_i \rangle\}$ is weakly C^∞-convergent to a stable (J, H)-map $\langle f_\infty \rangle$ if there are $f_i \in \langle f_i \rangle, f_\infty \in \langle f_\infty \rangle$ such that the following conditions hold.

(i) After stabilized by adding minimal number of markings, the stabilized domains $\Sigma_i = \Sigma_{f_i}$ is convergent to $\Sigma_\infty = \Sigma_{f_\infty}$ in the sense that when i is large enough, there exist identifications of stable $\mathcal{F}$-curves, $\phi_i : \Sigma_{(u_i, v_i)} \to \Sigma_i$ and $\phi_\infty : \Sigma_{(0,0)} \to \Sigma_\infty$, such that $(u_i, v_i) \to (0, 0)$ as $i \to \infty$. Here $\Sigma_{(u_i, v_i)}$ is the local deformation of $\Sigma_{(0,0)}$ defined before in Section 2.1.

(ii) Given any compact subset $K \subset \Sigma_{(0,0)} \setminus \{\text{singular points}\}$, there is an obvious embedding $\imath_i^K : K \to \Sigma_{(u_i, v_i)}$ through the gluing construction, when i is large. Define $f_i^K = f_i \circ \phi_i \circ \imath_i^K : K \to V$ and $f_\infty^K = (f_\infty \circ \phi_\infty)|_K$. We require that $\{f_i^K\}_{i=1}^\infty$ is C^∞-convergent to f_∞^K for any K as above.

(iii) $\lim_i E(f_i) = E(f_\infty)$.

We will call the induced topology on $\mathcal{FM}(J, H, A)$ and $\mathcal{GM}(J, H, A)$ the weak C^∞-topology.

Theorem 3.1 *$\mathcal{FM}(J, H, A)$ and $\mathcal{GM}(J, H, A)$ are compact and Hausdorff with respect to the weak C^∞-topology.*

The compactness part of this theorem for cuspidal maps is known as Gromov-Floer compactness theorem. The analysis there can be adapted here to prove the corresponding part of our theorem up to some suitable modification. The Hausdorffness is not true for the moduli space of cuspidal maps, but only holds

for the moduli space of stable (J, H)-maps. The complete proof of this statement is in [LiuT1], Sec 4. We refer our readers to the proof there.

To define the strong L_k^p-topology we mentioned before, we need to work with stable L_k^p-maps.

• Strong L_k^p-topology and local uniformizer

• Local deformation of stable (J, H)-maps.

We start with defining the local deformation of a stable (J, H)-map. Again we only deal with stable maps with $\mathcal{F}$-curves as domains. Given a stable map $\langle f \rangle$, let $f \in \langle f \rangle$ be a representative with $\mathcal{F}$-curve (Σ, l) as its domain. Let $(\Sigma_{(u,v)}, l)$ be the local universal deformation. We define $F_{(u,0)} : \Sigma_{(u,0)} \to V$ and $f_{(u,v)} : \Sigma_{(u,v)} \to V$ as follows.

Choose a family of homomorphisms $\phi_{(u,0)}$ of $\Sigma_{(u,0)}$ to $\Sigma_{(0,0)}$ such that the restriction of $\phi_{(u,0)}$ to each component of $\Sigma_{(u,0)}$ is a diffeomorphism and it maps all double points on the components of $\Sigma_{(u,0)}$ to the corresponding double points of $\Sigma_{(0,0)}$. Moreover, $\phi_{(u,0)}$ is identity on each component of $\Sigma_{(u,0)}$ outside a prescribed small neighborhood of its double points. When $|u|$ is small enough, such a $\phi_{(u,0)}$ exists. Note that $\phi_{(u,0)}$ is not holomorphic.

We define that $f_{(u,0)} = f \circ \phi_{(u,0)}$.

Now $f_{(u,v)}$ is obtained from $f_{(u,0)}$ by the following gluing procedure with gluing parameter v.

It is sufficient to consider the following two simplest cases:

(i) $f_{(u,0)} = f_1 \cup f_2$ with f_1 being a principal component and f_2 being a bubble component. Let $d_1 = d_2$ be their double points, associated with a complex gluing parameter t.

(ii) $f_{(u,0)} = f_1 \cup f_2$ with both of them being principal components jointed at their double point z (one of their "ends"). Associate with z a positive real gluing parameter τ.

The case of gluing two bubble components is the same as case (i) above and the general case can be reduced to above cases.

For case (i), let D_1 and D_2 be the small discs of Σ_1 and Σ_2 centered at d_1 and d_2 respectively. Let $(s_i, \theta_i), i = 1, 2$ be their cylindrical coordinates given by $w_i = e^{-(s_i + i\theta_i)}$. Then $\Sigma_{(u,t)}$ is obtained from $\Sigma_{(u,0)} = \Sigma_1 \cup \Sigma_2$ by cutting off $\{(s_i, \theta_i) \mid s_i > -\log|t|\} \hookrightarrow D_i$ and gluing back to remaining part of Σ along the boundaries through a rotation of $arg\, t$. Choose a cut-off function

$$\beta(s) = \begin{cases} 1 & s < -\log|t| - 2 \\ 0 & s > -\log|t| - 1. \end{cases}$$

We define

$$f_{(u,t)}(w) = \begin{cases} f_i(w), & w \in \Sigma_i \setminus \{(s_i, \theta_i) | s_i < -\log|t| - 2\} \\ Exp_{f(d)} \beta(s_i) \cdot \xi_i(w), & w \in \{(s_i, \theta_i) | s_i > -\log|t| - 2\}, \end{cases}$$

where $\xi_i(w)$ is defined by $f_i(w) = Exp_{f(d)} \xi_i(w)$ when $|w|$ is small.

Case (ii) can be treated in a similar way. We leave it to the readers.

• Local uniformizer and L_k^p-topology of $\mathcal{FB}(A)$.

From now on, we will assume that $\dim V \geq 4$. Let (J,H) be a generic pair. Under the assumption , it is proved in [FHS]

Theorem 3.2 *Given $f \in \mathcal{FM}^D(c_-, c_+, J, H, A)$, we have either*
(i) f is θ-independent and hence a gradient line of ∇H or
(ii) there exists an integer $m > 1$, such that $f = \tilde{f} \circ \pi_m$, where $\pi_m : \mathbf{R}^1 \times S^1 \to \mathbf{R}^1 \times S^1$ is given by $(s,\theta) \to (ms, m\theta)$ and $\tilde{f}$ is simple in the sense that there is no further factorization. Moreover if f is already simple, there exists at least one point $(s_0,\theta_0) \in \mathbf{R}^1 \times S^1$ such that $f(s,\theta) \neq f(s_0,\theta_0)$ if $(s,\theta) \neq (s_0,\theta_0)$ and Rank$(df_{(s_0,\theta_0)}) = 2$. We will call such point (s_0,θ_0) an injective point. Note that $\tilde{f} \in \mathcal{FM}^D(c_-, c_+, J, \frac{1}{m}H, \frac{1}{m}A)$.

As pointed out in [FHS], if the theorem holds for (J,H), so does it for $(J, \frac{1}{m}H)$.

Given $\langle f \rangle \in \mathcal{FM}(J,H,A)$, choose a representative $f \in \langle f \rangle$. If an unstable principal component f_i^P is θ-dependent, it covers a simple map $\tilde{f}_i^P$. We may assume that there is an injective point (s_i,θ_i) of $\tilde{f}_i^P$ lying on the middle circle $\{s_i = 0\}$.

For simplicity, we may assume that $(s_i,\theta_i) = (0,0) \in l_i$ and use $y_i = (s_i,\theta_i)$ to stabilize f_i^P. For any unstable bubble component f_j^B, it follows from [M] that, similar to the theorem above, $f_j^B = \tilde{f}_j^B \circ \pi_j$ such that $\tilde{f}_j^B$ has only injective points away from finite points of B_j and $\pi_j : B_j \to S_2$ is a finite branch covering. Choose $y_j^k, 1 \leq k \leq 2$ to stabilize B_j in such a way that $\pi_j(y_j^k)$ is an injective point.

Let $\tilde{\mathbf{H}}_i$ be the local hypersurface of codimension two at $f_i^P(y_i)$ such that f_i^P is transversal to $\tilde{\mathbf{H}}_i$ at y_i when f_i^P is not θ-independent. We then choose a local hypersurface $\mathbf{H}_i$ of codimension one such that $\tilde{\mathbf{H}}_i \hookrightarrow \mathbf{H}_i$ and $f_i^P|_{l_i}$ is transversal to $\mathbf{H}_i$ at y_i. When f_i^P is θ-independent, simply choose $\mathbf{H}_i$ of codimension one such that $\mathbf{H}_i$ is transversal to f_i^P at $y_i = (0,0)$. Similarly for each unstable bubble f_j^B, choose hypersurface $\tilde{\mathbf{H}}_j^k, 1 \leq k \leq 2$, of codimension 2 such that f_j^B is transversal to $\tilde{\mathbf{H}}_j^k$ at y_j^k. Let $\mathbf{H}$ be the collection of all those hypersurfaces $\mathbf{H}_i$'s and $\tilde{\mathbf{H}}_j^k$'s. Consider the local deformation $f_{(u,v)}$ of f with $\|(u,v)\| < \delta$ for some fixed small $\delta > 0$. Choose an $\epsilon > 0$, we define a local uniformizer of $\mathcal{FB}(A)$ near $\langle f \rangle \in \mathcal{FM}(J,H,A)$,

$$\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H}) =$$
$$\{g = g_{(u,v)} \,|\, \|g_{(u,v)} - f_{(u,v)}\|_{k,p} < \epsilon, g_{(u,v)}(y_i) \in \mathbf{H}_i,\ g_{(u,v)}(y_j^k) \in \tilde{\mathbf{H}}_j^k\},$$

where $g_{(u,v)} : \Sigma_{(u,v)} \to V$ and $y_i, y_j^k \in \Sigma_{(u,v)}$ through gluing. Here the metric on $\Sigma_{(u,v)}$ to define the L_k^p-norm is induced from that of $\Sigma_{(0,0)}$ through gluing.

Before we state any properties of $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$, we define the L_k^p-topology on $\mathcal{FB}(A)$ and $\mathcal{GB}(A)$ by using a similar construction as above. We only treat the

case $\mathcal{FB}(A)$ as before. Given $\langle f\rangle \in \mathcal{FB}(A)$, choose a representative $f \in \langle f\rangle$ and define

$$\mathcal{F}\tilde{U}_\epsilon(f) = \{g = g_{(u,v)}\,|\,\|(u,v)\| < \epsilon,\ \|g_{(u,v)} - f_{(u,v)}\| < \epsilon\}.$$

The process of forgetting those markings $y_i, y_j^k \in \Sigma_{(u,v)}$ induces a natural projection map

$$\pi_{\mathcal{F}} = \pi_{\mathcal{F}}(f) : \mathcal{F}\tilde{U}_\epsilon(f) \to \mathcal{F}U_\epsilon(f) = \pi_{\mathcal{F}}(\mathcal{F}\tilde{U}_\epsilon(f)) \hookrightarrow \mathcal{FB}(A).$$

Let $\mathcal{F}U = \{\mathcal{F}U_\epsilon(f)\,|\, f \in \langle f\rangle,\ \langle f\rangle \in \mathcal{FB}(A)\}$. One can directly check that

Lemma 3.4 *$\mathcal{F}U$ form a topological basis on $\mathcal{FB}(A)$.*

We will call the induced topology the (strong) L_k^p-topology on $\mathcal{FB}(A)$. In particular we get an induced strong L_k^p-topology on $\mathcal{FM}(J,H,A)$ as a subspace of $\mathcal{FB}(A)$. It is proved in [LiuT1], Section 4

Theorem 3.3 *The two topologies on $\mathcal{FM}(J,H,A)$ ($\mathcal{GM}(J,H,A)$) are equivalent. In particular, $\mathcal{FM}(J,H,A)$ ($\mathcal{GM}(J,H,A)$) is also compact with respect to L_k^p-topology.*

One of the corollary of this equivalence is

Corollary 3.1 *There exists an open neighborhood W of $\mathcal{FM}(J,H,A)$ ($\mathcal{GM}(J,H,A)$) in $\mathcal{FB}(A)$ ($\mathcal{GB}(A)$) such that W is Hausdorff with respect to the L_k^p-topology.*

We leave its proof to our readers as it will not be used in the rest of this paper. Now we come back to $\mathcal{F}U_\epsilon(f;\mathbf{H})$.

Definition 3.3

$$\Gamma_f = \{\phi|\phi : \Sigma_f \to \Sigma_f \textit{ is an automorphism, } f\phi = f\}.$$

Here each ϕ is a self identification of Σ_f as an n-pointed $\mathcal{F}$-curve .

Since each constant component of f is stable, it follows from the existence of injective points for simple maps that Γ_f is finite.

The next lemma explains why $\mathcal{F}U_\epsilon(f;\mathbf{H})$ forms a local uniformizer of $\mathcal{FB}(A)$.

Lemma 3.5 *When ϵ is small enough, there exists a continuous right action of Γ_f on $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$, which is smooth on each open strata of $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$. The natural projection $\pi_{\mathcal{F}} : \mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H}) \to \mathcal{FB}(A)$ commutes with Γ_f. The induced quotient map $\bar{\pi}_{\mathcal{F}} : \mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})/\Gamma_f \to \mathcal{FB}(A)$ gives rise to a homomorphism of $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})/\Gamma_f$ and an open neighborhood of $\langle f\rangle$ in $\mathcal{FB}(A)$.*

Remark 3.1 *Here the topology on $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$ is the L^p_k-topology, which can be defined similar to what we did for $\mathcal{FB}(A)$. The smooth structure for each strata $\mathcal{F}\tilde{U}^D_\epsilon(f;\mathbf{H})$ is the obvious one induced from the corresponding Banach manifold of product of mapping spaces.*

Proof:
Define

$$\tilde{\Gamma}_f = \{\phi \,\|\, \phi : \Sigma_f \to \Sigma_f \text{ is an automorphism. } f \circ \phi \in \mathcal{F}U_\epsilon(f, \mathbf{H})\}.$$

We prove first that when $\epsilon > 0$ is small enough, $\tilde{\Gamma}_f = \Gamma_f$. In fact let $y_f = \{f^{-1}(f(y_i)), f^{-1}(f(y^k_j))\}$ be the collection of the inverse images of the images of those markings added for stabilizing Σ_f. y_f is a finite set and $\tilde{\Gamma}_f$ is a subgroup of the permutation group $Sym(y_f)$. Note that $f^{-1}(f(y_i)) = y_i$. Hence $\phi(y_i) = y_i, \phi \in \tilde{\Gamma}_f$. But the elements of $\tilde{\Gamma}_f$ may permute different bubble components which lie on a same principal component. Let $m = \min_{\phi \in Sym(y_f)}\{\|f - f \circ \phi\| > 0\}$. It is easy to see that when $0 < \epsilon << m$ we have $\Gamma_f = \tilde{\Gamma}_f$.

Given $\phi \in \Gamma_f$ and $g \in \mathcal{F}U_\epsilon(f;\mathbf{H})$ with $g = g_{(u,v)} : \Sigma_{(u,v)} \to V$. We want to define the right action $g * \phi$. When ϵ is small enough, $g_{(u,v)}$ and $f_{(u,v)}$ are C^1-close to each other. This implies that near $\phi(y_i) = y_i$ and $\phi(y^k_j)$ of $\Sigma_{(u,v)}$ there exist points $y_i(\phi, g)$ and $y^k_j(\phi, g)$ uniquely determined by g such that $g(y_i(\phi, g)) \in \mathbf{H}_i$ and $g(y^k_j(\phi, g)) \in \mathbf{H}^k_j$. Note that here $\phi(y_i)$ and $\phi(y^k_j)$ come from the corresponding points in $\Sigma = \phi(\Sigma)$ through gluing.

Now consider stable $\mathcal{F}$-curve $\Sigma = \phi(\Sigma)$ equipped with markings ($\phi(y_i)$, $\phi(y^k_j)$, x). There exists a marking preserving identification

$$\psi_{(u'',v'')} : (\phi(\Sigma)_{(u'',v'')}; \phi(y_i), \phi(y^k_j), x) \to (\Sigma_{(u,v)}; y_i(g, \phi), g^k_j(g, \phi), x)$$

for some gluing parameter (u'', v''). Clearly, there is also an identification induced by ϕ,

$$\phi_{(u',v')} : (\Sigma_{(u',v')}, y_i, y^k_j, x) \to (\phi(\Sigma)_{(u'',v'')}; \phi(y_i), \phi(y^k_j), x)$$

for some (u', v'). Now we define

$$g * \phi = g \circ \psi_{(u'',v'')} \circ \phi_{(u',v')} : \Sigma_{(u'v,v')} \to V.$$

Now we prove that for $g_1, g_2 \in \mathcal{F}U_\epsilon(f, \mathbf{H})$, $\langle g_1 \rangle = \langle g_2 \rangle \iff \exists \phi \in \Gamma_f$ such that $g_1 = g_2 * \phi$.

We only need to prove the $\Longrightarrow$ part.

Suppose $\langle g_1 \rangle = \langle g_2 \rangle$ with $g_i : \Sigma_{(u_i,v_i)} \to V$, $i = 1, 2$. Then there exists an identification of $\mathcal{F}$-curves $\tilde{\phi} : \Sigma_{(u_1,v_1)} \to \Sigma_{(u_2,v_2)}$, which preserves the fixed marked points x and marked lines but may not preserve those y's, such that $g_1 = g_2 * \tilde{\phi}$. Let $(y_f)_{(u_i,v_i)} \hookrightarrow \Sigma_{(u_i,v_i)}, i = 1, 2$ be the finite subset of $\Sigma_{(u_i,v_i)}$

corresponding to y_f of Σ_f through gluing. Consider $\tilde{\phi}(y_{(u_1,v_1)}) \hookrightarrow \Sigma_{(u_2,v_2)}$. Then for each element in $\tilde{\phi}(y_{(u_1,v_1)})$, there is a unique element in $(y_f)_{(u_2,v_2)}$ such that the elements of $\tilde{\phi}(y)$ lie in a small disc centered at the corresponding element of $(y_f)_{(u_2,v_2)}$. This induces an injective map from $\tilde{\phi}(y)$ to $(y_f)_{(u_2,v_2)}$, hence an injective map $\tilde{\phi}_y : y \to y_f$. Now both g_1 and g_2 are in $\mathcal{F}\tilde{U}_\epsilon(f, \mathbf{H})$. When $\epsilon << m$, this can happen only if $\tilde{\phi}_y$ is induced from some $\phi \in \Gamma_f$. Having obtained such a ϕ, it is easy to see that $g_1 = g_2 * \phi$.

Similarly, we can prove that for any $g \in \mathcal{F}\tilde{U}_\delta(f)$, when $\delta << \epsilon$, there exists some $\tilde{g} \in \mathcal{F}U_\epsilon(f, \mathbf{H})$ such that $\langle g \rangle = \langle \tilde{g} \rangle$.

□

• **Orbifold bundles**

• Orbifold structure of $\mathcal{FB}(A)$ near $\mathcal{FM}(J, H, A)$.

Let $\mathcal{F}U_\epsilon(f, \mathbf{H}) = \pi_{\mathcal{F}}(\mathcal{F}\tilde{U}_\epsilon(f, \mathbf{H}))$, then $\mathcal{F}U_\epsilon(f, \mathbf{H})$ is an open neighborhood of $\langle f \rangle \in \mathcal{FB}(A)$. Consider the open covering

$$\mathcal{FM}(J, H, A) \hookrightarrow \bigcup_{\langle f \rangle \in \mathcal{FM}(J,H,A)} \mathcal{F}U_\epsilon(f, \mathbf{H}).$$

The compactness of $\mathcal{FM}(J, H, A)$ with respect to the (strong) L^p_k-topology implies that there exist finite $f_i, i = 1, \cdots, m$, such that

$$\mathcal{FM}(J, H, A) \hookrightarrow \bigcup_{i=1}^{m} \mathcal{F}U_{\epsilon_i}(f_i, \mathbf{H}_i).$$

Now we use U_i to denote $\mathcal{F}U_{\epsilon_i}(f_i, \mathbf{H}_i)$ and $\tilde{U}_i$ to denote its uniformizer. Let $U = \sum_{i=1}^m U_i$.

Theorem 3.4 *U has a stratified orbifold structure with respect to the local uniformizers.*

Proof:

The proof is a routine verification of the definition of orbifolds. We only indicate the main step and leave the details to our readers.

We only need to prove that if $\langle g \rangle \in U_1 \cap U_2$ with U_i being uniformized by $\tilde{U}_i$ with automorphism group Γ_i, $i = 1, 2$, then there exists an open neighborhood U of $\langle g \rangle$, such that (i) $U \hookrightarrow U_1 \cap U_2$; (ii) U is uniformized by $\tilde{U}$ with automorphism group Γ such that there exist two injection homomorphisms $\imath_i : \Gamma \to \Gamma_i$ and two (Γ, Γ_i)-equivariant embeddings $\lambda_i : \tilde{U} \to \tilde{U}_i$, $i = 1, 2$. Here the equivariant condition means that for any $\phi \in \Gamma, h \in \tilde{U}$, $\lambda_i(h * \phi) = \lambda_i(h) * \imath_i(\phi), i = 1, 2$.

In our case, let

$$g^i = g^i_{(u_i,v_i)} \in \langle g \rangle$$

be the representatives in $\tilde{U}_i = \mathcal{F}\tilde{U}_\epsilon(f_i, \mathbf{H}_i), i = 1, 2$. Then $g^i(y^i) \in \mathbf{H}_i, i = 1, 2$. Here we have used y^i to denote the collection of all those $y's$ in $\Sigma_{(u_i,v_i)}$ through

the gluing, which are originally in Σ^i and are used to stabilize the unstable components of Σ^i.

Now define

$$\tilde{V}_i = \{g = g_{(u,v)} \, \|g_{(u,v)} - g^i_{(u,v)}\| < \delta_i, g_{(u,v)} \in \tilde{U}_i|$$

and $\tilde{\Gamma}_i = \{\phi, | \, \phi \in \Gamma_i, g^i * \phi = g^i\}$.

It is easy to see that when $\delta_i << \epsilon_i, i = 1, 2$, $(\pi_i)_{\mathcal{F}} : \tilde{V}_i/\tilde{\Gamma}_i \to \mathcal{FB}(A)$ is an embedding onto some open neighborhood $V_i \hookrightarrow U_1 \cap U_2$. We may assume that $V_1 = V_2 = W$. Therefore we get two uniformizer $(\tilde{V}_i, \tilde{\Gamma}_i)$ of W. Clearly the inclusion map $(\tilde{V}_i, \tilde{\Gamma}_i) \to (\tilde{U}_i, \Gamma_i)$ gives rise to an injective $(\tilde{\Gamma}_i, \Gamma_i)$-equivariant map. The theorem is valid if we can prove that $(\tilde{V}_1, \tilde{\Gamma}_1)$ and $(\tilde{V}_2, \tilde{\Gamma}_2)$ are equivalent as uniformizers. Now choose a forgetting marking process for $\Sigma^i_{(u_i,v_i)}$ deleting out all extra markings in y^i needed to make $\Sigma^i_{(u_i,v_i)}$ stable. Let $\tilde{y}^i$ be the remaining markings in $\Sigma^i_{(u_i,v_i)}$ and $\tilde{\mathbf{H}}^i$ be the corresponding collection of local hypersurfaces. Form $\mathcal{F}\tilde{U}_{\delta_i}(\tilde{g}^i, \tilde{\mathbf{H}}^i)$ and denote them by $\tilde{W}_i$, where $\tilde{g}^i$ is same as g^i as a map but its domain has on extra marking anymore. One can directly check tha there is an equivariant embeding of $\tilde{W}_i$ into an open subset of $\tilde{V}_i$. Now since $\tilde{g}^i$ has no extra markings after the forgetting marking process and $\langle \tilde{g}_1 \rangle = \langle \tilde{g}_2 \rangle$, we can easily construct an equivariant equivalence of $\tilde{W}_1$ and $\tilde{W}_2$.

□

• Local T^{N_P}-action on $\mathcal{F}\tilde{U}^D_\epsilon(f, \mathbf{H})$.

Let D be an intersection pattern with N_P principal components. We now define a local $(S^1)^{N_P}$-action on $\mathcal{F}\tilde{U}^D_\epsilon(f; \mathbf{H})$. We will call such an action a (local) toric T^{N_P}-action. Given $g \in \mathcal{F}\tilde{U}^D_\epsilon(f; \mathbf{H})$, let $g = \bigcup_{i=1}^{N_P}\{g^P_i \cup_k g^B_{i,k}\}$, where $g^B_{i,k}$ are the bubble components lying above g^P_i. For any

$$\phi = (\phi_1, \cdots, \phi_{N_P}) \in (S^1)^{N_P},$$

with $|\phi|$ small, we will define $g * \phi$ by defining the action ϕ_i on $g^P_i \cup_k g^B_{i,k}, i = 1, \cdots, N_P$. For this purpose we only need to know how to define the action for $\phi \in S^1$, with $|\phi|$ small, on a stable map $g \in \mathcal{F}\tilde{U}^D(f; \mathbf{H})$ in the following two simplest cases:

(i) $g = g^P \cup g^B$.

Let $\Sigma = \Sigma_g = (P, l_P, d_P) \cup (B, y_1, y_2, d_B)$, where y_1, y_2 are marked points added for stabilizing B and $d_P = d_B$ is the double point of Σ. Note that $g^B(y_i) \in \mathbf{H}_i, i = 1, 2$. Choose an identification $(P; l_P) \cong (\mathbf{R}^1 \times S^1, \{\theta = 0\})$ so that, under such an identification, R_ϕ of rotation of ϕ-angle of P is well-defined. Now the domain

$$\Sigma^\phi = \Sigma_{g*\phi} = (P; l_P, R^{-1}_\phi(d_P)) \bigcup (B; y_1, y_2, d_B)$$

with double point $R^{-1}_\phi(d_P) = d_B$.

Let $I_\phi : \Sigma^\phi \to \Sigma$ be given by $R_\phi : P \to P$ and $Id : B \to B$. Note that I_ϕ does not preserve the marked lines l_P. We define $g * \phi = g \circ I_\phi$.

(ii) $g = g^P$ and the domain of g has only one unstable principal component $\Sigma = (P; l_p, y), y \in l_P$ and $g(y) \in \mathbf{H}$. Consider $g|_{R_\phi(l_P)} : R_\phi(l_P) \to V$. When $|\phi|, \epsilon$ are small enough. There is a unique $y_\phi \in I_\delta(R_\phi(y))$ for some given $\delta > 0$ such that $g(y_\phi) \in \mathbf{H}$. Let T_ϕ be the s-translation sending $R_\phi(y)$ to y_ϕ. We define $g * \phi : (P; l_P, y) \to V$ given by $g * \phi = g \circ T_\phi \circ R_\phi$.

We summarize the properties of the T^{N_P}-action in the next lemma.

Lemma 3.6 *For any intersection pattern D of N_P principal components, there exists a local T^{N_P}-action on $\mathcal{F}\tilde{U}^D(f;\mathbf{H})$, which is smooth in the variable of $\mathcal{F}\tilde{U}^D(f;\mathbf{H})$ and C^l-smooth in the variable of T^{N_P}, where $l = [k - \frac{2}{p}]$. Let $\mathcal{F}\tilde{U}^{\bar{D}}(f;\mathbf{H})$ be the union of all $\mathcal{F}\tilde{U}^{D_1}(f;\mathbf{H})$ with $D_1 \leq D$. Then the toric T^{N_P}-action has a continuous extension to $\mathcal{F}\tilde{U}^{\bar{D}}(f;\mathbf{H})$. The action is free , when all $f_i^P, i = 1, \cdots, N_P$, is θ-dependent. Moreover, the action of Γ_f commutes with the local toric action.*

Proof:

We only need to prove the statement concerning free action. It follows from our assumption that when $|u|$ is small, each principal component of $f_{(u,0)}$ contain at least one point such that $f_{(u,0)}$ is a local embedding near that point.When ϵ is small enough, so does for any

$$g \in \mathcal{F}\tilde{U}^D(f;\mathbf{H}).$$

The desired conclusion follows from this.

$\square$

It follows from this theorem that there is always an S^1-action on $\mathcal{F}\tilde{U}^{\bar{D}}(f;\mathbf{H})$ via the diagonal map from S^1 to T^{N_P}. There is an obvious way to extend the S^1-action to $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$, as we did for extending the Γ_f-action. Since we will describe a similar process in next section, we refer readers to there for this.

• Orbifold bundle $(\mathcal{L}, W)$.

This is an infinite dimensional bundle $\mathcal{L}$ over $\mathcal{FB}(A)$ defined as follows.

For any $\langle f \rangle \in \mathcal{FB}(A)$, we define

$$\mathcal{L}_{\langle f \rangle} = \bigcup_{f \in \langle f \rangle} \mathcal{L}^p_{k-1}(\wedge^{0,1}(f^*TV))/\sim,$$

where the equivalence relation $\sim$ is defined via pull-back of the sections induced from the identification of the domains.

Over $W \hookrightarrow \mathcal{FB}(A)$, $\mathcal{L}$ has an orbifold bundle structure. In fact, over each uniformizer $\tilde{U}_i = \mathcal{F}\tilde{U}_{\epsilon_i}(f_i;\mathbf{H}_i)$, there is a bundle $\tilde{L}_i \to \tilde{U}_i$, which form a uniformizer of $\mathcal{L}|_{U_i}$ with covering group Γ_i. For any $g \in \tilde{U}_i$, we define

$$(\tilde{\mathcal{L}}_i)_{(g)} = L^p_{k-1}(\wedge^{0,1}(g^*TV)).$$

The action of Γ_i on $\tilde{U}_i$ lifts to $\tilde{\mathcal{L}}_i$ via pull-back. Similarly the local T^{N_P}-action on $\tilde{U}_i^D$ also lifts to $\tilde{\mathcal{L}}_i^D$ in the same way.

The topology and smooth structure of $(\mathcal{L}, W)$ can be described as follows.

Fix a gluing parameter (u,v), we use $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$ to denote

$$\{g = g_{(u,v)} \,|\, \|g_{(u,v)} - f_{(u,v)}\| < \epsilon,\ g(y) \in \mathbf{H}\}.$$

Let $\tilde{\mathcal{L}}^{(u,v)}$ be the restriction of $\tilde{\mathcal{L}}$ to $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$. Then $\tilde{\mathcal{L}}^{(u,v)}$ has a local trivialization over $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$ induced from a J-invariant parallel transformation of (V,J) (see for example, [M] and [F1]). This bundle structure for $\tilde{\mathcal{L}}^{(u,v)}$ gives rise to a smooth structure for $\tilde{\mathcal{L}}^{(u,v)}$.

To define topology for $\mathcal{L}_W$, it is sufficient to define it for $\tilde{\mathcal{L}} \to \tilde{U}$ as we did for $\mathcal{FB}(A)$. Given $\xi \in (\tilde{\mathcal{L}})_g, g \in \tilde{U}$, we define $\xi_{(u,v)} \in (\tilde{\mathcal{L}})_{g_{(u,v)}}$ for small $\|(u,v)\|$ as follows.

Without loss of generality, we may assume that g has only two components $(\Sigma_i, d_i), i = 1,2$, with only one double point $d = d_1 = d_2$. Then we define $\xi_{(u,0)} = \xi_{(0,0)} = \xi$. Let $D_\delta(d_i) = \{w_i \,||w| < \delta\}$ be the δ-disc of Σ_i centered at d_i with complex coordinate w_i. We use (s_i, θ_i) to denote the corresponding cylindrical coordinate. Choose a cut-off function

$$\beta(s) = \begin{cases} 1 & s < -1 \\ 0 & s > 1. \end{cases}$$

Note that over $D_\delta(d)$, $g_{(u,v)}(s,\theta) = g_{(u,0)}(s,\theta)$ if $s < -\log|v| - 2$. We define

$$\xi_{(u,v)}(s,\theta) = \beta(s_1 + \log|v|) \cdot \xi^1_{(u,0)}(s_1,\theta_1) + (1 - \beta(s_1 + \log|v|))\xi^2_{(u,0)}(s_2,\theta_2).$$

We now define an ϵ-neighborhood of ξ in $\tilde{\mathcal{L}}$ by first define

$$\tilde{U}_\epsilon^{(u,v)}(\xi) = \{\eta = \eta_{(u,v)} \,|\, \eta_{(u,v)} \in \mathcal{L}^p_{k-1}(g^*_{(u,v)}, TV),\ \|\eta_{(u,v)} - \xi_{(u,v)}\|_{k-1,p} < \epsilon\}$$

for each fixed (u,v). Then for each (u,v), using the parallel transformation to move $\tilde{U}_\epsilon^{(u,v)}(\xi)$ to the fiber of $\mathcal{L}^{(u,v)}$ over $h_{(u,v)}$, with $\|h_{(u,v)} - g_{(u,v)}\| < \epsilon$. We use $\tilde{U}_\epsilon(\xi)$ to denote the collection of all images of $\tilde{U}_\epsilon^{(u,v)}(\xi)$ under the parallel transformation. The collection of all $\tilde{U}_\epsilon(\xi)$ form a base of a topology on $\tilde{\mathcal{L}} \to \tilde{U}$.

Given the Hamiltonian function H, we can define a section $s^i_H : \tilde{U}_i \to \tilde{\mathcal{L}}_i$ for the bundle $\tilde{\mathcal{L}}_i$ as follows. For $f = f_j^P \cup f_k^B$, we define $s^i_H(f)|_{B_k} \equiv 0$ and

$$s^i_H(f)|_{P_j} = \nabla H \circ f_j^P ds - J\nabla H \circ f_j^P d\theta.$$

$\bar{\partial}_J$-operator also induces an obvious section on $\tilde{\mathcal{L}}_i \to \tilde{U}_i$ by sending $g \in \tilde{U}_i$ to $\bar{\partial}_J g = dg + J \circ dg \circ i$. We use $\bar{\partial}^i_{J,H}$ to denote $\bar{\partial}_J + s^i_H$.

We summarize what we have achieved this far in the following theorem.

Theorem 3.5 *There is a (stratified) orbifold bundle $\mathcal{L}$ over an open neighborhood W of $\mathcal{FM}(J,H,A)$ in $\mathcal{FB}(A)$. The orbifold structure on $\mathcal{L}^D$ is compatible with the local T^{N_P}-action. The Hamiltonian function H induces a Γ_i-equivariant continuous section $\bar{\partial}^i_{J,H}$ of $(\tilde{\mathcal{L}}_i, \tilde{U}_i)$, which is smooth over each $\tilde{U}_i^{(u,v)}$. Moreover when restricted to $(\tilde{\mathcal{L}}_i^D, \tilde{U}_i^D)$, $\bar{\partial}^i_{J,H}$ is local T^{N_P}-equivariant.*

4 T^{N_P}-equivariant obstruction sheaf (local theory)

In this section, we will construct an obstruction sheaf $\tilde{\mathcal{R}}_f$ over a T^{N_P}-invariant uniformizer $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H}$ of W, where $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H}$ is the 'completion' of $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H}$ so that the local T^{N_P} action can be extended into a global one. The space $\tilde{K}_f = \Gamma(\tilde{\mathcal{R}}_f)$ of the sections of $\mathcal{R}_f$ is a subspace of $\Gamma(\tilde{\mathcal{L}}, \mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H}))$. Our goal in this section is to prove Theorem 4.1, which claims that each element of $\tilde{K}_f$, when restricted to $\mathcal{F}\tilde{U}^D_\epsilon(f^e;\mathbf{H})$ is $(T)^{N_D}$-invariant, where N_D is the number of principal components of D.

Most of this section will be devoted to construct directly the section space $\tilde{K}_f$. Our results here can be expressed as a solution to an abstract extension problem.

For any $f \in \langle f\rangle \in \mathcal{FM}(J,H,A)$, let $(\tilde{\mathcal{L}}, \mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H}))$ be a local uniformizer with a local T^{N_P} -action on each $(\tilde{\mathcal{L}}^D, \mathcal{F}\tilde{U}^D_\epsilon(f;\mathbf{H}))$. Fix a finite dimensional subspace $K \hookrightarrow \mathcal{L}_f = \{\xi \,|\, \xi \in \wedge^{0,1}(f^*T(V))\}$ with the following properties
(i) there exists a $\delta > 0$ such that for any element $\eta \in K, \eta_{D_\delta} = \eta|_{D_\delta} = 0$, where D_δ is the union of all δ-discs centered at double points of Σ_f;
(ii) for any $\eta \in K$, $\eta|_{P_i} = 0$ if f_i^P is a θ-independent principal component.

The main question that we want to answer in this section is whether it is possible to extend each element $\eta \in K$ into a local section $\tilde{\eta}$ of the bundle $\tilde{\mathcal{L}} \to \mathcal{F}\tilde{U}^D_\epsilon(f;\mathbf{H})$ near f in such a way that
(a) $\tilde{\eta}$ is (locally) T^{N_P}-equivariant;
(b) $\tilde{\eta}$ is smooth on $\mathcal{F}\tilde{U}^{(u,v)}(f;\mathbf{H})$ for any fixed (u,v);
(c) $\tilde{\eta}$ is continuous.

One can also formulate the same question for $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H}$ with T^{N_P} action.

There is a very simple way to extend elements in $K \to \mathcal{L}_f$ locally. The vanishing property $\eta_{D_\delta} = 0$ for any $\eta \in K$ implies that η can be extended over the local deformation $f_{(u,v)}$, for small $\|(u,v)\|$, in an obvious way. We then use parallel transformation to extend it to $\mathcal{F}\tilde{U}^{(u,v)}(f;\mathbf{H})$, for fixed (u,v). Let η^e be the resulting extension of η. We will call it the canonical extension of η. In general η^e may not be T^{N_P}-equivariant. One may try to use the usual averaging process to make η^e into a T^{N_P}-equivariant section. However there are two obvious difficulties that make it impossible to directly use this usual averaging process. First of all our T^{N_P}-action is only locally defined. Secondly, even in the case that the usual averaging process is applicable to η^e, the resulting

T^{P_N}-equivariant section may not be equal to η at f. In fact it may happen that it is even not close to η at f. In this case , the transversality argument needed for gluing may fail.

To see the nature of difficulties better, we give a different description of $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$.

We define a new uniformizer $\mathcal{F}\tilde{U}_\epsilon(f;\tilde{\mathbf{H}})$. If there is no unstable θ-dependent principal component of f, $\mathcal{F}\tilde{U}_\epsilon(f;\tilde{\mathbf{H}})$ is just $\mathcal{F}\tilde{U}_\epsilon(f:\mathbf{H})$. Otherwise, we may assume that $f_i^P, i=1,\cdots,N_P^1$ are the θ-independent unstable principal components and $f_m^P, m=N_P^1+1,\cdots,N_P^2$ are the θ-dependent unstable principal components. Then for each such θ-dependent principal component f_m^P, there exists a local hypersurface $\tilde{\mathbf{H}}_m$ at $f_m^P((0,0))$ of codimension two transversal to f_m^P at $(0,0)$. Now for each such unstable component P_m, we will also allow y_m, the marking used for stabilizing P_m, to vary in $\{s=0\}$. This will introduce a new real parameter θ_m for the local deformation of f and Σ_f. It describes the θ-coordinate for y_m. Let $\tilde{u}$ be the collection of the parameters in u together with θ_m's. We now define the local deformation $f_{\tilde{u}}=f\circ R_{\tilde{u}}$, where $R_{\tilde{u}}$ is the rotation which brings θ_m to $(0,0)$ on each P_m.

Let $D=D_f$. We set

$$\begin{aligned}&\mathcal{F}\tilde{U}^D_\epsilon(f,\tilde{\mathbf{H}})=\\&\{g=g_{\tilde{u}},\||g_{\tilde{u}}-f_{\tilde{u}}\|_{k,p}<\epsilon, g_i^P(y_i)\in\mathbf{H}_i, g_m^P(y_m)\in\tilde{\mathbf{H}}_m, g_j^B(y_j^k)\in\tilde{\mathbf{H}}_j^k\}.\end{aligned}$$

We can define $\mathcal{F}\tilde{U}_\epsilon(f,\tilde{\mathbf{H}})$ similarly by using the local deformation $f_{(\tilde{u},v)}$, where $f_{(\tilde{u},v)}$ is obtained from $f_{\tilde{u}}$ by the gluing with gluing parameter v.

We now define the corresponding T^{N_P}-action. We start with the action on $\mathcal{F}\tilde{U}^D_\epsilon(f,\tilde{\mathbf{H}})$. Clearly, we only need to define $g_m^P*\phi$, for $g\in\mathcal{F}\tilde{U}^D_\epsilon(f,\tilde{\mathbf{H}})$ and $\phi\in S^1$. The domain of $g_m^P*\phi$ is $(P_m,l_m,R_\phi^{-1}(y_m))$, and we define $g_m*\phi=g_m^P\circ R_\phi$.

We now extend the induced S^1-action to $\mathcal{F}\tilde{U}_\epsilon(f,\tilde{\mathbf{H}})$. Given $g=g_{(\tilde{u},v)}$ and $\phi\in S^1$, we define the ϕ-rotation of the principal components, still denoted as R_ϕ,

$$R_\phi:(\Sigma_{(\tilde{u},v)};y_i,R_\phi^{-1}(y_m),y_j^k)\to(\Sigma_{(\tilde{u},v)};R_\phi(y_i),y_m,y_j^k),$$

which is a rotation of $\arg\phi$ on each principal component with respect to marked line there and is identity on each bubble component. Consider $g\circ R_\phi$. There are unique $y_i'\in I_\delta(y_i), y_m'\in D_\delta(R_\phi^{-1}(y_m))$ and $y_j^{k'}\in D_\delta(y_j^k)$ such that $g\circ R_\phi(y_i')\in\mathbf{H}_i$, $g\circ R_\phi(y_m')\in\tilde{\mathbf{H}}_m$ and $g\circ R_\phi(y_j^k)\in\tilde{\mathbf{H}}_j^k$. Then there is a unique new parameter $(\tilde{u}',v')$ such that

$$(\Sigma_{(\tilde{u}',v')};y_i,R_\phi^{-1}(y_m),y_j^k)\cong(\Sigma_{(\tilde{u},v)},y_i',y_m',y_j^{k'})$$

under the identification map $\psi:\Sigma_{(\tilde{u}',v')}\to\Sigma_{(\tilde{u},v)}$. We define $g*\phi=g\circ R_\phi\circ\psi$.

We remark that one can use a similar construction to extend the local S^1-action on $\mathcal{F}\tilde{U}^D_\epsilon(f,\mathbf{H})$ defined in previous section to $\mathcal{F}\tilde{U}_\epsilon(f,\mathbf{H})$.

Now observe that the action above is well-defined for all $\phi \in T^{N_P}$, not just locally for $|\phi|$ small. This suggests us to enlarge $\mathcal{F}\tilde{U}_\epsilon(f, \mathbf{H})$ so that T^{N_P}-action can be defined globally. For this purpose, we fix a large integer M. For each element $I = (i_1, \cdots, i_{N_P}) \in (\mathbf{Z}/M\mathbf{Z})^{N_P}$, set

$$\phi_I = (\phi_{i_1}, \cdots, \phi_{i_{N_P}})$$

where $\phi_{i_k} = \frac{2\pi i_k}{M}$. We define f_I and $\mathcal{F}\tilde{U}_\epsilon(f_I, \mathbf{H})$ as follows. f_I is same as f when restricted to any stable principal components or bubble components. When restricted to unstable principal component P_m, the domain of f_I is $(P_m, l_m, R^{-1}_{\phi_{i_m}}(y_m))$. Note that here the marked point $R^{-1}_{\phi_{i_m}}(y_m)$ used for stabilizing P_m for f_I lies on $R^{-1}_{\phi_{i_m}}(l_m)$. We define f_I to be $f \circ R_{\phi_{i_m}}$ on P_m. Now we can define $\mathcal{F}\tilde{U}_\epsilon(f_I, \mathbf{H})$ and local T^{N_P}-action on it by the very same formula as we did for $\mathcal{F}\tilde{U}_\epsilon(f, \mathbf{H})$.

Now form the disjoint union

$$\coprod_I \mathcal{F}\tilde{U}_\epsilon(f_I, \mathbf{H}), I \in (\mathbf{Z}/M\mathbf{Z})^{N_P}.$$

We introduce an equivalence relation between the elements in $\mathcal{F}\tilde{U}_\epsilon(f_I, \mathbf{H})$ and elements in its neighbors. A typical neighbor has a form $\mathcal{F}\tilde{U}_\epsilon(f_J, \mathbf{H})$ with all $j_k = i_k, k \neq l$ and $j_l = i_l + 1$ for some $1 \leq l \leq N_P$. Without loss of generality, we may assume that $l = 1$, $I = (i, \hat{I})$, $J = (i+1, \hat{I})$. Given such I and J there is a coordinate change $T_{I,J}$ from $\mathcal{F}\tilde{U}_\epsilon(f_I, \mathbf{H})$ to $\mathcal{F}\tilde{U}_\epsilon(f_J, \mathbf{H})$ as follows. For any $g \in \mathcal{F}\tilde{U}^D_\epsilon(f_I, \mathbf{H})$, $D = D_f$, we define $T_{I,J}(g) = g$ along all stable principal components, bubble components and those unstable principal components $g^P_l, l > 1$. Now consider the unstable principal component g^P_1. Since $g^P_1(R^{-1}_{\phi_i}(y_1)) \in \mathbf{H}_1$, there exists a unique y'_{i+1} lying on $R^{-1}_{\phi_{i+1}}(l_1)$ and near $R^{-1}_{\phi_{i+1}}(y_1)$ such that $g^P_1(y'_{i+1}) \in \mathbf{H}_1$, when g is close enough to the local deformation of f_I and M is large enough. Let T_r be the s-translation for P_1 that brings $R^{-1}_{\phi_{i+1}}(y_1)$ to y'_{i+1}. We define $T_{I,J}$ along g^P_1 to be $g^P_1 \circ T_r$. One can easily extend $T_{I,J}$ from fixed intersection pattern D to the general case by using a similar process of extending Γ_f-action before.

Let $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$ denote the above disjoint union quotienting out the equivalence relation introduced by these $T_{I,J}$'s.

Lemma 4.1 *$\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$ is a (stratified) Banach manifold. The local T^{N_P}-action on $\mathcal{F}\tilde{U}_\epsilon(f_I; \mathbf{H})$ are compatible with the "coordinate changing" maps $T_{I,J}$, $I, J \in (\mathbf{Z}/M\mathbf{Z})^{N_P}$. This defines a local T^{N_P}-action on $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$, which can be extended into a (global) T^{N_P}-action.*

Proof:

One can directly check that the local T^{N_P}-action on each coordinate chart $\mathcal{F}\tilde{U}_\epsilon(f_I; \mathbf{H})$ is preserved under coordinate changes $T_{I,J}$. The usual process to

complete a local action to a global one for a compact Lie group is applicable here, which yields a well-defined T^{N_P}-action.

$\square$

Lemma 4.2 *There is a T^{N_P}-equivariant equivalence map*

$$\rho : \mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H}) \to \mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}}).$$

Proof:

We only need to define

$$\rho^D : \mathcal{F}\tilde{U}^D_\epsilon(f^e; \mathbf{H}) \to \mathcal{F}\tilde{U}^D_\epsilon(f; \tilde{\mathbf{H}})$$

with $D = D_f$. The extension from ρ^D to ρ is a routine procedure. We omit it here.

Without loss of generality, we may assume that f contains some unstable principal component f^P_m, $m < N_P$, which is θ-dependent. Given $g \in \mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$, we may assume that $g \in \mathcal{F}\tilde{U}_\epsilon(f; \mathbf{H}) = \mathcal{F}\tilde{U}_\epsilon(f_0; \mathbf{H})$. Clearly, we only need to define ρ^D along those unstable principal component g^P_m. There exists a unique point $\tilde{y}'_m$ near y_m in the domain P_m such that $g^P_m(\tilde{y}'_m) \in \tilde{\mathbf{H}}_m \hookrightarrow \mathbf{H}_m$.

We define the domain of the m-th unstable principal component of $\rho(g)$ to be $(P_m, l_m, \tilde{y}_m)$, where $\tilde{y}_m$ has same θ-coordinate as $\tilde{y}'_m$ has and lies on th central circle $\{s = 0\}$. Now we define

$$(\rho(g))^P_m = g^P_m \circ Tr^m : (P_m, l_m, \tilde{y}_m) \to V,$$

where Tr^m is the s-translation of P_m that brings $\tilde{y}_m$ to $\tilde{y}'_m$. One can directly check that the above definition of ρ, which is given by using particular coordinate of $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$ is actually compatible with coordinate changing maps $T_{I,J}$. A direct calculation shows that ρ is one-to-one and commutes with the T^{N_P}-actions defined on the corresponding spaces.

$\square$

Because of this lemma, we may use $\mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$ to replace $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$ for our problem of finding a T^{N_P}-equivariant extension of K.

Theorem 4.1 *There exists an extension of K over $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$, which has the property described at the beginning of this section.*

Proof:

We only need to prove the corresponding statement for $\mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$. The following two cases are to be considered.
(i) All unstable principal components f^P_m are θ-dependent; (ii) some of the unstable principal components are θ-independent.

Case (i): Recall that in this case there is another kind of local deformation $f_{(\alpha,\tilde{t})}$. The domain $\Sigma_{(\alpha,\tilde{t})}$ of $f_{(\alpha,\tilde{t})}$ is obtained from $\Sigma = \Sigma_f$, with the gluing

parameter $(\alpha, \tilde{t})$, where α is the first component of $u = (\alpha, \theta)$ and $\tilde{t}$ is obtained from $v = (t, \tau)$ by adding a rotational component to τ (see Section 2.1). $f_{(\alpha,\tilde{t})}$ is obtained from $f_{(\alpha,0)}$ through a similar gluing process to the previous one with the gluing parameter $\tilde{t}$. We define

$$\mathcal{G}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}}) = \{g = g_{(\alpha,\tilde{t})} \mid |g_{(\alpha,\tilde{t})} - f_{(\alpha,\tilde{t})}| < \epsilon, g(y_i) \in \tilde{H}_i, g(y_j^k) \in \tilde{H}_j^k\}.$$

There is an obvious projection map

$$p_f : \mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}}) \to \mathcal{G}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}}),$$

given by simply forgetting the marked lines of the elements of $\mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$. Let $\mathcal{L}^{\mathcal{F}}$ and $\mathcal{L}^{\mathcal{G}}$ be the corresponding bundles over the above two spaces. Clearly, $(p_f)^*(\mathcal{L}^{\mathcal{G}}) = \mathcal{L}^{\mathcal{F}}$. We may identify $K \hookrightarrow \Gamma(\mathcal{L}_f^{\mathcal{F}})$ with a subspace of $\Gamma(\mathcal{L}_f^{\mathcal{G}})$. Let K' be the corresponding subspace. Suppose that $\dim K = k$ and $\{e_1, \cdots, e_k\}$ is a basis for K. Let $e_i', i = 1, \cdots, k$ be the corresponding elements in K' with $p_f^*(e_i') = e_i$.

Since each e_i' vanishes near all double points of f, we can easily get a canonical extension $(e_i')^c$ of a section of the bundle $\mathcal{L}^{\mathcal{G}}$ over $\mathcal{G}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$ as we did for stable maps with $\mathcal{F}$-curves as domains described before in this section. We define $\tilde{e}_i = p_f^*((e_i')^c)$.

Now for any $\phi \in T^{N_P(g)}$ and $g \in \mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$, $p_f(g * \phi) = p_f(g)$. This implies that $\tilde{e}_i, i = 1, \cdots, k$, has the required invariant property.

Case (ii):

We may assume in this case that $f_i^P, i = 1, \cdots, N_P^1$ be the θ-independent unstable principal components and $f_m^P, m = N_P^1 + 1, \cdots, N_P^2$ be the other unstable principal components. Note that in this case $N_P^1 < N_P$.

We now define a local deformation of f, which is a mixture of those deformations as stable $\mathcal{F}$-maps and stable $\mathcal{G}$-maps. The idea is to deform the θ-independent part of unstable principal components as stable $\mathcal{F}$-maps and the rest as stable $\mathcal{G}$-maps. When the topological type is fixed, the domain of such a deformation is described by the parameter α appeared in the corresponding deformation as stable $\mathcal{G}$-maps, since in this case all those markings y_m of P_m, $m \leq N_P^1$, are fixed. The gluing parameters that control the topological type of the deformation can be described as follows.

On $\mathcal{F}$-curve part and $\mathcal{G}$-curve part of Σ_f, we use the usual gluing parameter respectively. That is we associate the gluing parameter $\tau = (\tau_2, \cdots, \tau_{N_P^1})$ to the ends $z_2, \cdots, z_{N_P^1}$ and $\tilde{t}$ to those double points and "ends" in $\mathcal{G}$-curve part. Now the key point is to associate the "ends" $z_{N_P^1+1}$, the double point that divides Σ_f into the two parts, with a complex gluing parameter w. We will use $\tilde{t}'$ to denote $(\tau, w, \tilde{t})$. Set $\alpha' = \alpha$. We have the deformation $\Sigma_{(\alpha', \tilde{t}')}$ of Σ. The deformation of $f_{(\alpha',0)}$ and $f_{(\alpha',\tilde{t}')}$ can be defined in a similar way as before.

We now define

$$\mathcal{H}\tilde{U}_\epsilon(f;\tilde{\mathbf{H}}) = \left\{ g_{(\alpha',\tilde{t}')} \;\middle|\; \begin{array}{cc} |g_{(\alpha',\tilde{t}')} - f_{(\alpha',\tilde{t}')}| < \epsilon, & g_{(\alpha',\tilde{t}')}(y_j^k) \in \tilde{\mathbf{H}}_j^k, \\ g_{(\alpha',\tilde{t}')}(y_i) \in \mathbf{H}_i, & g_{(\alpha',\tilde{t}')}(y_m) \in \tilde{\mathbf{H}}_m, \end{array} \right\},$$

where $y_i \in P_i$ of a θ-independent unstabl principal component, $y_m \in P_m$ of a θ-dependent unstabl principal componen and $y_j^k \in B_j^k$ of an unstable bubble component.

Fix an intersection pattern D. We have the following two cases:

(a) the gluing parameter w determined by D is zero.

In this case, we may assume that each element g in the strata has $N_P(D_1)$ those principal components, whose domains are obtained from the domains of θ-independent unstable principal components of f through gluing. Let $N_P(D_2)$ be the number of the other principal components of g. We decompose $\mathcal{F}\tilde{U}_\epsilon^D(f;\tilde{\mathbf{H}})$ as the product,

$$\mathcal{F}\tilde{U}_\epsilon^{D_1}(f;\tilde{\mathbf{H}}) \times \mathcal{F}\tilde{U}_\epsilon^{D_2}(f;\tilde{\mathbf{H}}),$$

where the first factor contains those $g_i^P, 1 \le i \le N_P(D_1)$ and the second contains all the other components of g. Similarly, we can decompose $\mathcal{H}\tilde{U}_\epsilon^D(f;\tilde{\mathbf{H}})$ as the product, $\mathcal{H}\tilde{U}_\epsilon^{D_1}(f;\tilde{\mathbf{H}}) \times \mathcal{H}\tilde{U}_\epsilon^{D_2}(f;\tilde{\mathbf{H}})$. Let $p_i : \mathcal{F}\tilde{U}_\epsilon^D(f;\tilde{\mathbf{H}}) \to \mathcal{F}\tilde{U}_\epsilon^{D_i}(f;\tilde{\mathbf{H}}), i = 1, 2$, be the projection. Then we define

$$p_f^D(g) = (p_1(g), p_f^{D_2} p_2(g)),$$

where

$$p_f^{D_2} : \mathcal{F}\tilde{U}_\epsilon^{D_2}(f;\tilde{\mathbf{H}}) \to \mathcal{H}\tilde{U}_\epsilon^{D_2}(f;\tilde{\mathbf{H}})$$

is defined as in case (i). Clearly p_f^D commutes with the $T^{N_P(D_2)}$-action on the second factor.

(b) $w \neq 0$. In this case, we need to add one more factor $\mathcal{F}\tilde{U}_\epsilon^{D_3}(f;\tilde{\mathbf{H}})$ to the above decomposition of $\mathcal{F}\tilde{U}_\epsilon^D(f;\tilde{\mathbf{H}})$, which corresponds to the principal component of g "passing through" w together with all bubbles lying on this component. Let $\mathcal{H}\tilde{U}_\epsilon^{D_i}(f;\tilde{\mathbf{H}}), i = 1, 2, 3$, be the corresponding decomposition of $\mathcal{H}\tilde{U}_\epsilon^D(f;\tilde{\mathbf{H}})$. We define $p_i, i = 1, 2, 3$ similarly. Now one can directly verify that $\mathcal{F}\tilde{U}_\epsilon^{D_3}(f;\tilde{\mathbf{H}})$ and $\mathcal{H}\tilde{U}_\epsilon^{D_3}(f;\tilde{\mathbf{H}})$ are homeomorphic to each other(in the case $w \neq 0$) and diffeomorphic when restricted to subspace with fixed deformation type. Let

$$p_f^{D_3} : \mathcal{F}\tilde{U}_\epsilon^{D_3}(f;\tilde{\mathbf{H}}) \to \mathcal{H}\tilde{U}_\epsilon^{D_3}(f;\tilde{\mathbf{H}})$$

be the corresponding homeomorphism. We define

$$p_f^D(g) = (p_1(g), p_f^{D_2} \cdot p_2(g), p_f^{D_3}(p_3(g))).$$

Again p_f^D commutes with the $T^{N_P(D_2)}$-action, where $N_P(D_2)$ is the number of principal components in $\mathcal{F}\tilde{U}_\epsilon^{D_2}(f;\tilde{\mathbf{H}})$. One can prove that all these p_f^D's, pasted together, define a continuous map

$$p_f : \mathcal{F}\tilde{U}_\epsilon(f;\tilde{\mathbf{H}}) \to \mathcal{H}\tilde{U}_\epsilon(f;\tilde{\mathbf{H}}),$$

which is smooth when restricted to each subspace of fixed deformation type.

Now as in case (i), we can easily get a canonical extension $\tilde{e}'_i, i = 1, \cdots, k$, of e'_i where $\tilde{e}'_i$ is a section of the bundle $\mathcal{L}^{\mathcal{H}}$ over $\mathcal{H}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$. We then consider $\tilde{b}_i = p_f^*(\tilde{e}'_i)$ of the corresponding section of $\mathcal{L}^{\mathcal{H}}$ over $\mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$. Unlike the case (i), in this case, $\tilde{b}_i$ does not have the required invariant property. In fact, given $g \in \mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$, if $D = D_g$ is in the case (a) above, then $\tilde{b}_i$ is already $T^{N_P(D)}$-invariant and we simply define $\tilde{e}_i^D = \tilde{b}_i^D$ over $\mathcal{F}\tilde{U}_\epsilon^D(f; \tilde{\mathbf{H}})$, since in this case, p_f^D commutes with the $T^{N_P(D_2)}$-action acting on $\mathcal{F}\tilde{U}_\epsilon^{D_2}(f; \tilde{\mathbf{H}})$ and b_i can be chosen to be zero along those elements of $\mathcal{F}\tilde{U}_\epsilon^{D_1}(f; \tilde{\mathbf{H}})$ by our assumption on K. In the case that D is in case (b), $\tilde{b}_i^D$ so defined only has $T^{N_P(D_1)+N_P(D_2)}$-invariant. There is an extra S^1-action coming from the "rotations" of elements of $\mathcal{F}\tilde{U}_\epsilon^{D_3}(f; \tilde{\mathbf{H}})$. Given $g \in \mathcal{F}\tilde{U}_\epsilon^D(f; \tilde{\mathbf{H}})$ and $\phi \in S^1$, we use

$$A_\phi^D : \mathcal{F}\tilde{U}_\epsilon^D(f; \tilde{\mathbf{H}}) \to \mathcal{F}\tilde{U}_\epsilon^D(f; \tilde{\mathbf{H}})$$

to denote this action for fixed $\phi \in S^1$. We define

$$\tilde{e}_i^D = \frac{1}{2\pi} \int_{S^1} A_\phi^{D*}(\tilde{b}_i^D) d\phi.$$

It follows from the definition that $\tilde{e}_i^D$ is $T^{N_P(D)}$-invariant. One can directly verify that $\tilde{e}_i^D$ is compatible to each other when D varies and hence gives rise to a well-defined continuous extension $\tilde{e}_i$ of e_i with all the required properties of the theorem.

We note that in the case (ii) above, we have assumed that the first N_P^1 principal components of f are unstable and θ-independent. This assumption simplifies the way of choosing gluing parameter $\tilde{t}'$ of $f_{(\alpha', \tilde{t}')}$ and other related constructions. The general case can be treated in a similar manner, but with more complicated notations.

Now let $\tilde{K} = span\{\tilde{e}_1, \cdots, \tilde{e}_k\}$.

□

Lemma 4.3 *When p is even, there exists a T^{N_P}-invariant continuous cut-off function $\tilde{\beta}$ on $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$ with $0 \leq \tilde{\beta}(g) \leq 1$ such that $\tilde{\beta}(g) = 1$ in a neighborhood of the T^{N_P}-orbit of f and $\tilde{\beta} = 0$ near the boundary of $\mathcal{F}\tilde{U}_\epsilon(f^e; \mathbf{H})$. Moreover, $\tilde{\beta}$ is smooth on $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f^e; \mathbf{H})$.*

Proof:

Consider

$$p_f : \mathcal{F}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}}) \to \mathcal{H}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$$

in Theorem 4.1. Suppose that we can construct a continuous cut-off function $\beta_1 : \mathcal{H}\tilde{U}_\epsilon(f; \mathbf{H}) \to [0, 1]$ such that $\beta_1 = 1$ in a neighborhood of $p_f(f)$ and $\beta = 0$ near the boundary of $\mathcal{H}\tilde{U}_\epsilon(f; \tilde{\mathbf{H}})$. Repeating the process of finding $\tilde{e}_i$

with the required smoothness in previous Theorem, we set $\tilde{\beta}_1 = p_f(\beta_1)$ and $\tilde{\beta}^D = \frac{1}{2\pi}\int (A^D_\phi)^*(\tilde{\beta}^D_1)d\phi$. As before, $\tilde{\beta}^D$ will be pasted together to get a T^{N_P}-invariant cut-off function $\tilde{\beta}$ with the desired property. To construct β_1, we note that when p is even, the function $\rho^g_{(u,v)}(h) = \|h-g\|^p_{k,p}$ is a smooth function on $\mathcal{F}\tilde{U}^{(u,v)}_\epsilon(f;\mathbf{H})$. Using this we can easily construct the desired β_1.

□

Corollary 4.1 *In Theorem 4.1, we may choose $\tilde{K}$ in such a way that all its elements vanish near the boundary of $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H})$.*

5 Transversality and Gluing

In this section we will establish the transversality of perturbed $\bar{\partial}_{J,H,\nu}$-operation over a T^{N_P}-invariant uniformizer $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H})$, where the perturbation term ν is a generic element of $\tilde{K}$. Therefore $\bar{\partial}^\nu_{J,H,}$ is a T^{N_P}-equivariant transversal section of $\mathcal{L}$. Its zero set $\mathcal{F}\tilde{\mathcal{M}}^\nu(f^e)$ is a cornered smooth manifold of dimension $Ind(c_+) - Ind(c_-) + 2c_1(A) + 1$, with a T^{N_P}-action acting on it.

Our method in this section is an adaption of the method in [LiuT1] (see also [L1]). In this section often we will only quote results in [LiuT1] and indicate necessary changes to incorporate the T^{N_P}-action here. We refer reader for the detailed proof in [LiuT1].

5.1 Transversality

We start with giving a local coordinate charts for $\mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H})$ near f and local trivialization of $\mathcal{L}$ over those coordinate charts.

Let $y_i \in P_i$ and $y^k_j \in B_j, 1 \le k \le 2$, be the marked points added to Σ_f for stabilizing it, and $\mathbf{H}_i$ and $\tilde{\mathbf{H}}^k_j$ be the local hypersurfaces at $f^P_i(y_i)$ and $f^B_j(y^k_j)$ used before for slicing.

For each $\mathbf{H}_i$ and $\tilde{\mathbf{H}}^k_j$, let $h_i = T_{f(y_i)}\mathbf{H}_i$, $h^k_j = T_{f(y^k_j)}\tilde{\mathbf{H}}^k_j$. We may assume that both $\mathbf{H}_i$ and $\tilde{\mathbf{H}}^k_j$ are totally geodesic so that they are the local images of h_i and h^k_j under the exponential map.

We define

$$L^p_k(f^*TV, h) = \{\xi, \,|\, \xi \in L^p_k(f^*TV),\, \xi(y_i) \in h_i, \xi(y^k_j) \in h^k_j\},$$

where the values of ξ from different components are the same at double points.

Similarly, we define $L^p_k(f^*_{(u,v)}TV, h)$.

Let

$$\tilde{V}^{(u,v)}_\epsilon = \{\xi \,|\xi \in L^p_k(f^*_{(u,v)}TV; h);\, \|\xi\|_{k,p} < \epsilon\},$$

and $\tilde{V}_\epsilon = \bigcup_{\|(u,v)\|<\delta} \tilde{V}^{(u,v)}_\epsilon$, which is a "bundle" over $\Lambda_\delta = \{(u,v) \,|\, \|(u,v)\| < \delta\,\}$.

The coordinate chart

$$Exp_f^{(u,v)} : \tilde{V}_\epsilon^{(u,v)} \to \mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$$

for $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$ is given by $\xi \to Exp_{f_{(u,v)}}\xi$.

The "coordinate chart" for $\mathcal{F}\tilde{U}_\epsilon(f;H)$ is given by

$$Exp_f = \bigcup_{(u,v)\in\Lambda_\delta} Exp_f^{(u,v)} : \tilde{V}_\epsilon = \bigcup_{(u,v)\in\Lambda_\delta} \tilde{V}_\epsilon^{(u,v)} \to \mathcal{F}\tilde{U}_\epsilon(f;\mathbf{H}) = \bigcup_{(u,v)\in\Lambda_\delta} \mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H}).$$

We already defined trivialization of $\mathcal{L}^{(u,v)}$ over $\mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H})$ by using the parallel transformation. Let

$$\psi^{(u,v)} : \mathcal{F}\tilde{U}_\epsilon^{(u,v)}(f;\mathbf{H}) \times L_{k-1}^p(\wedge^{0,1}(f_{(u,v)}^*TV)) \to \mathcal{L}^{(u,v)}$$

denote the trivialization here. Then

$$\gamma^{(u,v)} = \psi^{(u,v)} \circ (Exp_f^{(u,v)} \times Id) : \tilde{V}_\epsilon^{(u,v)} \times L_{k-1}^p(\wedge^{0,1}(f_{(u,v)}^*TV)) \to \tilde{\mathcal{L}}^{(u,v)}$$

gives rise to a trivialization of $\mathcal{L}^{(u,v)}$ in terms of above coordinate chart.

Let $\gamma = \cup_{(u,v)\in\Lambda_\delta}\gamma^{(u,v)}$. Then

$$\gamma : \bigcup V_\epsilon^{(u,v)} \times L_{k-1}^p(\wedge^{0,1}(f_{(u,v)}^*TV)) \to \tilde{\mathcal{L}}$$

is a local "trivialization" of $\tilde{\mathcal{L}}$.

Now under these local coordinate chart and local trivialization, the $\bar{\partial}_{J,H}$-section of $\tilde{\mathcal{L}}$ becomes:

$$F_{(u,v)}^1 = \pi_2 \circ (\gamma^{(u,v)})^{-1} \circ \bar{\partial}_{J,H} \circ Exp_f^{(u,v)} : V_\epsilon^{(u,v)} \to \mathcal{L}_{k-1}^p(\wedge^{0,1}(f_{(u,v)}^*TV)).$$

Let $F^1 = \bigcup_{(u,v)\in\Lambda_\delta} F_{(u,v)}^1$. Note that $F_{(u,v)}^1$ is smooth. Let

$$L_{(u,v)}^1 = (DF_{(u,v)}^1)_{f_{(u,v)}} : L_k^p(f_{(u,v)}^*TV;h) \to L_{k-1}^p(\wedge^{0,1}(f_{(u,v)}^*TV)).$$

Lemma 5.1 *Under our assumption that all critical points c_i of H are non-degenerate in the sense of Floer homology, $L_{(u,v)}^1$ is a Fredholm operator.*

In general we don't expect that $L_{(u,v)}^1$ is surjective, even for a generic choice of (J,H). Failure of the transversality by only perturbing the parameter (J,H) has been considered as a major difficulty in Floer homology and quantum cohomology. As we mentioned in the introduction of this paper, this difficulty had been overcome through the work [FO], [LiT] and [LiuT1]. Following the method we developed in [LiuT1], we define $K = K_f = coker L_{(0,0)}^1$, then

$$L_{(0,0)}^1 \oplus E : L_f^p(f^*TV;h) \oplus K \to L_{k-1}^p(f^*TV))$$

is surjective, where E is the inclusion. We can actually choose a modified K, still denoted as K, such that $L^1_{(0,0)} \oplus E$ is surjective and that K has the property described in the beginning of last section. As in there we extend K to $\tilde{K}$, whose elements are T^{N_P}-invariant section of $\mathcal{L}$ over $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H})$. Consider $\tilde{K}$ as a "bundle " over $\mathcal{F}\tilde{U}_\epsilon(f^e;\mathbf{H})$ with bundle projection $\pi_{\tilde{K}}$. Now we define $\bar{\partial}^{\tilde{K}}_{J,H} : \tilde{K} \to \tilde{\mathcal{L}}$ given by

$$\nu \to \bar{\partial}_{J,H}(\pi_{\tilde{K}}(\nu)) + \nu.$$

In terms of above local coordinate chart and local trivialization, each element $\tilde{\nu} \in \tilde{K}$ gives rise to a map :

$$V_\epsilon \to \bigcup_{(u,v)\in\Lambda_\delta} \mathcal{L}^p_{k-1}(\wedge^{0,1}(f^*_{(u,v)}TV))$$

and $\bar{\partial}^{\tilde{K}}_{J,H}$ becomes a function $F = \cup_{(u,v)\in\Lambda_\delta} F_{(u,v)}$, with

$$F_{(u,v)} = F^1_{(u,v)} \oplus \tilde{E} : V^{(u,v)}_\epsilon \oplus \tilde{K} \to \mathcal{L}^p_{k-1}(\wedge^{0,1}(f^*_{(u,v)}TV))$$

given by $(\xi, \nu) \to F^1_{(u,v)} + \nu(\xi)$. Clearly, $(DF_{(u,v)})_{(0,0)} = L^1_{(u,v)} + E_{(u,v)}$ where $E_{(u,v)} : K \to L^p_{k-1}(\wedge^{0,1}(f^*_{(u,v)}TV))$ is given by $e \to \tilde{e}|_{f_{(u,v)}}$. Let $L_{(u,v)} = L^1_{(u,v)} + E_{(u,v)}$. We know that $L_{(0,0)}$ is surjective. We want to prove that when δ is small enough, for any $(u, v) \in \Lambda_\delta$, $L_{(u,v)}$ is also surjective. In fact in order to do the gluing, we need somewhat more. We need to prove that when $\|(u, v)\|$ is small enough, $L_{(u,v)}$ has a uniformly right inverse with respect to some suitable exponential weighted norm on the domain and range of $L_{(u,v)}$. We will only consider the following simplest case for defining these norms, since this case already contains essential points of general case.

We assume that $f = f^P \cup f^B$ of two components with $(P, d_1) \cup (B, d_2)$ of double points $d_1 = d_2$. Let z_1, z_2 be the ends of P and y_1, y_2 be the marked points of B. Identify $D_{\epsilon_i}(d_i)$ with $\mathbf{R}^1 \times S^1 = \{(s_i, \theta_i)\}, i = 1, 2$ with d_i corresponding to $s_i = +\infty$. Those local deformations of f coming from only moving double point d along the central circle of P do not play any role in the following definitions. For the reason of simplicity, we omit them here. Therefore the local deformation f can be described by a single complex parameter $t \in D_\delta$. Given $\xi \in L^p_k(f^*_t TV; h)$, we define $\tilde{\xi}^0 = \int_{c_t} \xi|_{c_t} d\theta$, where $c_t = \{s_i = -\log|t|\}$ is the central circle of Σ_{f_t}.

Note that when $s_i > -\log|t|-1$, both $f^P(s_1, \theta_1)$ and $f^B(s_2, \theta_2)$ are just $f(d)$. Hence the above definition of $\tilde{\xi}^0$ makes sense and $\tilde{\xi}^0 \in T_{f(d)}V$. We may think $\tilde{\xi}^0$ as a vector field along f_t with $s_i > -\log|t_0|$ for some fixed t_0. Multiple $\tilde{\xi}^0$ with a fixed cut-off function in Σ_t, we extend $\tilde{\xi}^0$ to an element $\xi^0 \in L^p_k(f^*_t TV, h)$. Let $\xi^1 = \xi - \xi^0$.

Now for $\xi \in L^p_k(f^*_t TV, h)$, $\eta \in L^p_{k-1}(\wedge^{0,1}(f^*_t TV))$, we define $\|\xi\|_{k,p;\mu} = \|e^{\mu s}\xi^1\|_{k,p} + |\xi^0|$ and $\|\eta\|_{k-1,p;\mu} = \|e^{\mu s}\eta\|_{k-1,p}$, where $0 < \mu < 2\pi$ is fixed and $|\xi^0| = |\tilde{\xi}^0|$ is the Euclidean norm of $\xi^0 \in T_{f(d)}V$. For any element $(\xi, e) \in L^p_k(f^*_{(u,v)}TV, h) \oplus K$, we define $\|(\xi, e)\|_{k,p;\mu} = \|\xi\|_{k,p;\mu} + |e|$.

It is proved in [LiuT1] that

Proposition 5.1 *When $\|(u,v)\|$ is small enough, under these μ-exponential weighted norms*

$$L_{(u,v)} : L^p_k(f^*_{(u,v)}TV, h) \oplus K \to L^p_{k-1}(\wedge^{0,1}(f^*_{(u,v)}TV))$$

has a uniform right inverse $G_{(u,v)}$ in the sense that there exists a constant $C_1 = C_1(f)$, only depends on $f = f_{(0,0)}$ such that for any $\eta \in L^p_{k-1}(\wedge^{0,1}(f^*_{(u,v)}TV))$

$$\|G_{(u,v)}(\eta)\|_{k,p;\mu} \leq C_1\|\eta\|_{k-1,p;\mu}.$$

Corollary 5.1 *$L_{(u,v)}$ is surjective when $\|(u,v)\|$ is small.*

We will use $L^p_{k,\mu}(f^*_{(u,v)}TV, h)$ and $L^p_{k-1,\mu}(\wedge^{0,1}(f^*_{(u,v)}TV))$ to denote the corresponding spaces equipped with the μ-exponential weighted norms.

5.2 Gluing

Now a direct computation shows that the local deformation $f_{(u,v)}$ is an asymptotic solution of $\bar{\partial}_{J,H}g = 0$ when f is a stable (J,H)-map. More precisely, we have

Lemma 5.2

$$\lim_{(u,v)\to 0} \|\bar{\partial}_{J,H}f_{(u,v)}\|_{k-1,p;\mu} = 0.$$

To do gluing, we also need an estimate on the second order term $Q_{(u,v)}$ in the Taylor expansion of $F_{(u,v)}$:

$$V^{(u,v)}_\epsilon \subset L^p_{k,\mu}(f^*_{(u,v)}TV, h) \to L^p_{k-1,\mu}(\wedge^{0,1}(f^*_{(u,v)}TV)),$$

where $Q_{(u,v)}$ is defined by

$$F_{(u,v)}(\xi) = F_{(u,v)}(0) + L_{(u,v)}(\xi) + Q_{(u,v)}(\xi).$$

Lemma 5.3 *There exists a constant $C_2 = C_2(f)$ only depending on f such that for any $\xi_{(u,v)}$, $\eta_{(u,v)} \in L^p_{k,\mu}(f^*_{(u,v)}TV, h)$,*

$$(i) \quad \|Q(\xi_{(u,v)})\|_{k-1,p;\mu} \leq C_2\|\xi_{(u,v)}\|_\infty\|\xi\|_{k,p;\mu};$$

$$\begin{aligned} (ii) \qquad & \|Q(\xi_{(u,v)}) - Q(\eta_{(u,v)})\|_{k-1,p;\mu} \\ & \leq C_2(\|\xi_{(u,v)}\|_{k,p;\mu} + \|\eta_{(u,v)}\|_{k,p;\mu})\|\xi_{(u,v)} - \eta_{(u,v)}\|_{k,p;\mu}. \end{aligned}$$

Proof:

The corresponding statement was proved in [F1] when $k = 1$, and $1 - \frac{2}{p} > 0$. The general case here follows from that by a direct induction argument. $\Box$

Lemma 5.4 *(Picard method) Assume that a smooth map $F : X \to Y$ from Banach spaces $(X, \|\cdot\|)$ to Y has a Taylor expansion*

$$F(\xi) = F(0) + DF(0)\xi + Q(\xi)$$

such that $DF(0)$ has a finite dimensional kernel and a right inverse G satisfying

$$\|GQ(\xi) - GQ(\eta)\| \leq C(\|\xi\| + \|\eta\|)\|\xi - \eta\|$$

for some constant C. Let $\delta_1 = \frac{1}{8C}$. If $\|G \circ F(0)\| \leq \frac{\delta_1}{2}$, then the zero set of f in $B_{\delta_1} = \{\xi, \,|\, \|\xi\| < \delta_1\}$ is a smooth manifold of dimension equal to the dimension of $kerDF(0)$. In fact, if

$$K_{\delta_1} = \{\xi \,|\, \xi \in kerDF(0), \, \|\xi\| < \delta_1\}$$

and $K^{\perp} = G(Y)$, then there exists a smooth function

$$\phi : K_{\delta_1} \to K^{\perp}$$

such that $F(\xi + \phi(\xi)) = 0$ and all zeros of f in B_{δ_1} are of the form $\xi + \phi(\xi)$.

The proof of this Lemma is an elementary application of Banach's fixed point theorem (see [F1]). Now we apply the Picard method above to our case with

$$X = V_{\epsilon,\mu}^{(u,v)} \oplus \tilde{K}_{\epsilon} \hookrightarrow L^p_{k,\mu}(f^*_{(u,v)}TV, h) \oplus \tilde{K},$$

$$Y = L^p_{k-1,\mu}(\wedge^{0,1}(f^*_{(u,v)}TV)), \text{ and } F = F_{(u,v)}.$$

We have

Theorem 5.1 *When ϵ is small enough, the solution set $\mathcal{FM}^{K,(u,v)}_{\epsilon,\mu}(f)$ of the equation $F_{(u,v)} = 0$ in $V^{(u,v)}_{\epsilon,\mu} \times \tilde{K}_{\epsilon}$ is a smooth manifold of dimension $Ind(c_+) - Ind(c_-) + 2c_1(A) + 1 + r - 2n_{\alpha} - 2n_t - n_{\theta} - n_{\tau}$, where $r = \dim K$ and $n_{\alpha}, n_t, n_{\theta}$ and n_{τ} are the numbers of zero components in α, θ, t and τ respectively. Here $u = (\alpha, \theta), v = (t, \tau)$. Moreover $DF_{(u,v)}$ is surjective along the zero set $\mathcal{FM}^{K,(u,v)}_{\epsilon,\mu}(f)$.*

Note that the last statement follows from the fact that $F_{(u,v)}$ induces a local differentiable embedding:

$$F_{(u,v)} \oplus \pi_N : V^{(u,v)}_{\epsilon,\mu} \oplus K_{\epsilon} \to L^p_{k-1,\mu}(\wedge^{0,1}(f^*_{(u,v)}TV)) \oplus N^{(u,v)},$$

where $N^{(u,v)} = N$ is the kernel of $L_{(u,v)}$ and π_N is the projection with respect to the orthogonal decomposition $L^p_{k,\mu}(f^*_{(u,v)}TV, h) \oplus K = N \oplus N^{\perp}$. Here the L^2-inner product in the above decomposition is defined in terms of the "standard metric" on $\Sigma_{(u,v)}$ induced from $\Sigma_{(0,0)}$ through gluing.

The exponential weight μ-norm and the usual (k,p)-norm on $L^p_{k,\mu}(f^*_{(u,v)}TV, h)$ are not uniformly equivalent with respect to (u,v). However, it is proved in [LiuT1] that

Theorem 5.2 *When $\epsilon' << \epsilon$, the solution set $\{F_{(u,v)}(\xi)=0\}$ in $V^{(u,v)}_{\epsilon'} \times \tilde{K}_{\epsilon'}$ is contained in $\mathcal{F}\tilde{\mathcal{M}}^{K,(u,v)}_{\epsilon,\mu}(f)$.*

In other words, as far as the solution set are concerned, the exponential weight μ-norm and the usual (k,p)-norm are equivalent.

Because of this, we can reformulate Theorem5.1.

Theorem 5.3 *The solution set $\mathcal{F}\tilde{\mathcal{M}}^{K,(u,v)}_{\epsilon}(f)$ in $V^{(u,v)}_{\epsilon} \times \tilde{K}_{\epsilon}$ is a smooth manifold of* $\dim Ind(c_+) - Ind(c_-) + 2c_1(A) + 1 + r - 2(n_\alpha + n_t) - (n_\theta + n_\tau)$.

Let $N^{(u,v)}_\epsilon$ be the ϵ-ball in $N^{(u,v)}$ centered at origin. Let $N_\epsilon = \cup_{(u,v)\in\Lambda_\delta} N^{(u,v)}_\epsilon \cong \Lambda_\epsilon \times N^{(0,0)}_\epsilon$. There is a diffeomorphism of $T^{(u,v)} : N^{(u,v)}_\epsilon \to \mathcal{F}\tilde{\mathcal{M}}^{K(u,v)}_\epsilon(f)$ as described in Picard method.

Let

$$T = \bigcup_{(u,v)\in\Lambda_\delta} T^{(u,v)} : N_\epsilon \to \mathcal{F}\tilde{\mathcal{M}}^{K}_\epsilon(f) = \bigcup_{(u,v)\in\Lambda_\delta} \mathcal{F}\tilde{\mathcal{M}}^{K(u,v)}_\epsilon(f)$$

be the induced (continuous) identification. We give $\mathcal{F}\tilde{\mathcal{M}}^{K}_\epsilon(f)$ the (cornered) smooth structure of N_ϵ induced from T.

Theorem 5.4 *$\mathcal{F}\tilde{\mathcal{M}}^{K}_\epsilon(f)$ is a cornered smooth manifold of dimension $Ind(c_+) - ind(c_-) + 2c_1(A) + r + 1$. The smooth structure of $\mathcal{F}\tilde{\mathcal{M}}^{K}_\epsilon(f)$ is induced from $\mathcal{F}\tilde{\mathcal{M}}^{D_f,K}_\epsilon(f) \times \Lambda_\delta$ under the gluing map T.*

We now come to a T^{N_P}-invariant version of above theorem.

Let $\mathcal{F}\tilde{U}_\epsilon(f^e, \mathbf{H})$ be a T^{N_P}-invariant uniformizer near f. Then

$$\mathcal{F}\tilde{U}_\epsilon(f^e, \mathbf{H}) = \bigcup_I \mathcal{F}\tilde{U}_{\epsilon_I}(f_I, \mathbf{H}), I \in (\mathbf{Z}/M\mathbf{Z})^{N_P(D_f)},$$

where M is a fixed large integer. Note that the T^{N_P}-orbit of f is contained in $\mathcal{F}\tilde{U}_\epsilon(f^e, \mathbf{H})$. In particular, for each I, there exists a $\phi_I \in T^{N_P}$ such that $f_I = f * \phi_I$. Now assume that we have chosen a T^{N_P}-equivariant extension $\tilde{K}$ of K over $\mathcal{F}\tilde{U}_\epsilon(f^e, \mathbf{H})$. By exploring the naturality of all relevent construction, it is easy to see that if K-perturbed operator $\bar{\partial}^{\tilde{K}}_{J,H}$ is a transversal section in $\mathcal{F}\tilde{U}_{\epsilon_0}(f, \mathbf{H})$ for some sufficiently small ϵ_0, so is it on all $\mathcal{F}\tilde{U}_{\epsilon_I}(f_I, \mathbf{H})$, $I \in (\mathbf{Z}/M\mathbf{Z})^{N_P}$ for ϵ_I small enough. Because of the compactness of the T^N_P-orbit of f, we may choose M to be large enough so that $\bigcup_I \mathcal{F}\tilde{U}_{\epsilon_I}(f_I, \mathbf{H})$ already covers the orbit of f. We will still use $\mathcal{F}\tilde{U}_\epsilon(f^e, \mathbf{H})$ to denote this union. Now we have

Theorem 5.5 *When M is large enough and ϵ is small enough, the perturbed $\bar\partial^{\tilde K}_{J,H}$-operator is a transversal T^{N_P}-equivariant section of $\tilde{\mathcal{L}}$ over $\mathcal{F}\tilde U_\epsilon(f^e,\mathbf{H})\times \tilde K_\epsilon$. The solution set $\mathcal{F}\tilde{\mathcal{M}}^{\tilde K}_\epsilon(f^e)$ of $\bar\partial^{\tilde K}_{J,H}\xi=0$ is a cornered smooth manifold of dimension $Ind(c_+)-Ind(c_-)+2c_1(A)+r+1$. The cornered smooth structure of $\mathcal{F}\tilde{\mathcal{M}}^{\tilde K}_\epsilon(f^e)$ is obtained from the corresponding (cornered) smooth structure of $\mathcal{F}\tilde{\mathcal{M}}^{\tilde K,D_f}_\epsilon(f^e)\times\Lambda_\delta$ under the gluing map T.*

For any $\nu\in\tilde K_\epsilon$, let $\mathcal{F}\tilde{\mathcal{M}}^{\nu}_\epsilon(f^e)=\pi_2^{-1}(\nu)$, where $\pi_2:\mathcal{F}\tilde U_\epsilon(f^e,\mathbf{H})\times\tilde K_\epsilon\to\tilde K_\epsilon$ is the projection. We have

Theorem 5.6 *For a generic choice of $\nu\in\tilde K_\epsilon$, $\mathcal{F}\tilde{\mathcal{M}}^{\nu}_\epsilon(f^e)$ is a (cornered) smooth manifold of dimension $Ind(c_+)-Ind(c_-)+2c_1(A)+1$. There is a continuous T^{N_P}-action on $\mathcal{F}\tilde{\mathcal{M}}^{\nu}_\epsilon(f^e)$, which is smooth on each of its strata.*

6 T^{N_P}-invariant Virtual Moduli Cycles

In this section, we will globalize these T^{N_P}-invariant local moduli spaces $\mathcal{F}\tilde{\mathcal{M}}^{\nu_f}_\epsilon(f^e)$ to get a T^{N_P}-invariant virtual moduli cycles.

Note that we may cover the moduli space $\mathcal{F}\mathcal{M}(J,H,A)$ by using

$$\bigcup_{\langle f\rangle}\mathcal{F}U_{\epsilon_f}(f^e;\mathbf{H}_f),\langle f\rangle\in\mathcal{F}\mathcal{M}(J,H,A),$$

where

$$\mathcal{F}U_{\epsilon_f}(f^e,\mathbf{H}_f)=\pi_f(\mathcal{F}\tilde U_{\epsilon_f}(f^e,\mathbf{H}_f))$$

is the π_f-image of the T^{N_P}-invariant uniformizer $\mathcal{F}\tilde U_{\epsilon_f}(f^e,\mathbf{H})$. We may assume that there exist finite many f_i's, say, $i=1,\cdots,q$, such that (i) $\bigcup_{1\le i\le q}\mathcal{F}U_{\epsilon_i}(f^e_i;\mathbf{H}_i)$ already cover $\mathcal{F}\mathcal{M}(J,H,A)$; (ii) perturbed $\bar\partial^{\tilde K_i}_{J,H}$-operator is transversal to zero section on $\mathcal{F}\tilde U_{\epsilon_i}(f^e_i;\mathbf{H}_i)\times\tilde K_i$ and (iii) $\tilde K_i$ vanishes near the boundary of $\mathcal{F}\tilde U_{\epsilon_i}(f^e_i,\mathbf{H}_i)$.

We now use W_i to denote $\mathcal{F}U_{\epsilon_i}(f_i;\mathbf{H}_i)$ with a T^{N_P}-invariant uniformizer $\tilde W_i$ and cover group Γ_i. Let $\tilde{\mathcal{L}}_i$ be the corresponding bundle over $\tilde W_i$. Note that the Γ_i-action commutes with the action of T^{N_P}. This implies that the T^{N_P}-action descends to $W_i\hookrightarrow\mathcal{F}\mathcal{B}(A)$.

In order to globalize these perturbed moduli spaces $\mathcal{F}\tilde{\mathcal{M}}^{\nu_i}(f^e_i)$, we need to know how these perturbation terms ν_i change from $\tilde W_i$ to $\tilde W_j$. The idea now is to use a fiber product construction of the covering $(\tilde W_i,\pi_i)$ as a replacement of "intersections" of $\tilde W_i$'s.

For this purpose, let $\tilde{\mathcal{N}}$ be the nerve of the covering $W=\{W_i,1\le i\le q\}$. We will use elements of $\tilde{\mathcal{N}}$ as indices, In other words. we define the following multi-indices set

$$\mathcal{N}=\{I=(i_1,\cdots,i_n)\,|i_1<i_2<\cdots<i_n,W_{i_1}\cap W_{i_2}\cap\cdots W_{i_n}\neq 0\}.$$

We define the length $l(I) = n$, for $I = (i_1, \cdots, i_n)$. Note that $\mathcal{N}$ has an obvious partial order induced by inclusion. For any $I = (i_1, \cdots, i_n) \in \mathcal{N}$, we will use W_I to denote $W_{i_1} \cap W_{i_2} \cdots \cap W_{i_n}$, and $\mathcal{L}_I = \mathcal{L}|_{W_I}$.

There are n uniformizing systems

$$(\widetilde{\mathcal{L}}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}, \widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}; \pi_{i_1,\cdots,\widehat{i_k},\cdots,i_n})$$

of

$$(\mathcal{L}_I, W_I),$$

with covering group Γ_{i_k}, induced from

$$\pi_{i_k} : (\widetilde{\mathcal{L}}_{i_k}, \widetilde{W}_{i_k}) \to (\mathcal{L}_{i_k}, W_{i_k}),$$

where

$$\widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n} = (\pi_{i_k})^{-1}(W_I)$$

and

$$\widetilde{\mathcal{L}}_{i_1,\cdots,\widehat{i_k},\cdots,i_n} = \widetilde{\mathcal{L}}|_{\widetilde{W}_{i_1,\cdots,\widehat{i_k},\cdots,i_n}}.$$

We want to construct the pull-back of these morphisms, which is denoted by

$$\pi_I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}, \widetilde{W}_I^{\Gamma_I}) \to (\mathcal{L}_I, W_I)$$

with covering group

$$\Gamma_I = \Gamma_{i_1} \times \cdots \times \Gamma_{i_n}.$$

We define first

$$\widetilde{W}_I^{\Gamma_I} \quad = \quad \left\{ u \mid u \in \prod_{k=1}^{n} \widetilde{W}_{i_k}, \begin{array}{ccc} \pi_{i_k}(u_k) & \in & W_I, \\ \pi_{i_k}(u_k) & = & \pi_{i_l}(u_l) \end{array} \right\}.$$

We define π_I to be the composition of $\prod_{k=1}^{n} \pi_{i_k}$ restricting to $\widetilde{W}_I^{\Gamma_I}$ with $\triangle_n^{-1}$ of the inverse of n-fold diagonal. If $J = (j_1, \cdots, j_m) \subseteq I = (i_1, \cdots, i_n)$, there exists an obvious projection map

$$\pi_J^I : \widetilde{W}_I^{\Gamma_I} \to \widetilde{W}_J^{\Gamma_J}$$

induced from the corresponding projection $\prod_{i_k \in I} \widetilde{W}_{i_k}$ to $\prod_{j_l \in J} \widetilde{W}_{j_l}$ such that $\pi_J \circ \pi_J^I = \iota_J^I \circ \pi_I$ when restricted to the inverse image of π_J^I, where ι_J^I is the inclusion $W_I \hookrightarrow W_J$.

All the above constructions can be directly extended to bundle case and we get a system of bundles $\{p_I : \widetilde{\mathcal{L}}_I^{\Gamma_I} \to \widetilde{W}_I^{\Gamma_I}\}$, $I \in \mathcal{N}$.

Note that for any fixed I with $l(I) > 1$, $\widetilde{W}_I^{\Gamma_I}$ is not a (stratified) smooth manifold in general but rather a (stratified) smooth variety, i.e., locally it is a finite union of (stratified) smooth manifold. In fact for $u \in \widetilde{W}_I^{\Gamma_I}$ with

$u=(u_1,\cdots,u_n)$, $\bar u=\pi_{i_k}(u_k)$, we can choose an open neighborhood U of $\bar u$ in W_I and consider the inverse image $\widetilde U_k=\pi_{i_k}^{-1}(U)$ of u_k in $\widetilde W_{i_k}$. When U is small enough, there exist $(n-1)$ equivalence maps $\lambda_k:\widetilde U_1\to\widetilde U_k$, $k=2,\cdots,n$. Composing with the actions of automorphism group Γ_{u_k} of $\widetilde U_k$, we get $\prod_{i=2}^n|\Gamma_{u_i}|$ equivalence maps:

$$\phi_k\lambda_k:\widetilde U_1\to\widetilde U_k,\quad k=2,\cdots,n,\quad \phi_k\in\Gamma_{u_k}.$$

Clearly $u=(u_1,u_2,\cdots,u_n)\in\prod_{k=1}^n\widetilde U_k$ is contained in $\widetilde W_I^{\Gamma_I}$ if and only if $u_k=\phi_k\lambda_k(u_1)$ for some $\phi_k\in\Gamma_{u_k}$, $k>1$. Thus, in general we can identify a neighborhood $\widetilde U$ of u in $\widetilde W_I^{\Gamma_I}$ with an union of $\prod_{i\neq j}^n|\Gamma_{u_i}|$ copies of $\widetilde U_j$.

Note that the action of Γ_i commutes with T^{N_P}-action. That implies that all our constructions here are T^{N_P}-equivariant.

We summarize up the above discussion in the following lemma.

Lemma 6.1 *There exists a pull-back*

$$\pi_I:(\widetilde{\mathcal L}_I^{\Gamma_I},\widetilde W_I^{\Gamma_I})\to(\mathcal L_I,W_I)$$

of the n uniformizing systems

$$\pi_{i_1,\cdots,\widehat{i_k},\cdots,i_n}:(\widetilde{\mathcal L}_{i_1,\cdots,\widehat{i_k},\cdots,i_n},\widetilde W_{i_1,\cdots,\widehat{i_k},\cdots,i_n})\to(\mathcal L_I,W_I)$$

in the category of (stratified) smooth varieties with the automorphism group Γ_I.

For any $J\subset I$, there exists a projection

$$\pi_J^I:(\widetilde{\mathcal L}_I^{\Gamma_I},\widetilde W_I^{\Gamma_I})\to(\mathcal L_J^{\Gamma_J},W_J^{\Gamma_J}),$$

whose generic fiber contains $\frac{|\Gamma_I|}{|\Gamma_J|}$ points, where $\frac{|\Gamma_I|}{|\Gamma_J|}=\prod_{i_k\in I\setminus J}|\Gamma_{i_k}|$. It satisfies the relation that $\pi_J\circ\pi_J^I=\iota_J^I\circ\pi_I$ for each $I\in\mathcal N$, when restricted to the inverse image of π_J^I. Moreover, all constructions can be done in a T^{N_P}-equivariant manner.

Lemma 6.2 *There exists an open covering $\{V_I\}$, $I\in\mathcal N$ of $\mathcal{FM}(J,H;A)$ such that*

(i)$V_I\subset W_I$, for all $I\in\mathcal N$;

(ii) $Cl(V_{I_1})\cap Cl(V_{I_2})\neq\emptyset$ only if $I_1<I_2$, or $I_2<I_1$.

Moreover, all V_I can be chosen to be T^{N_P}-invariant.

Proof:

We may assume that there exist open sets $W_i^1\subset\subset W_i$, $i=1,\cdots,q$ such that $\{W_i^1,i=1,\cdots,q\}$ already forms a covering of $\mathcal{FM}(J,H,A)$. For each fixed i we can find pairs of open sets $W_i^j\subset\subset U_i^j$, $j=1,\cdots,q-1$ such that

$$W_i^1\subset\subset U_i^1\subset\subset W_i^2\subset\subset U_i^2\cdots\subset\subset W_i^q=W_i.$$

Now define

$$V_{i_1,\cdots,i_n} = W^n_{i_1} \cap W^n_{i_2} \cdots \cap W^n_{i_n} \setminus (\cup_{J\in\mathcal{N}_{n+1}} Cl(U^n_{j_1}) \cap Cl(U^n_{j_2}) \cdots \cap Cl(U^n_{j_{n+1}})),$$

where $J = (j_1, \cdots, j_{n+1})$.

Clearly the family $\{V_{i_1,\cdots,i_n}, (i_1, \cdots, i_n) \in \mathcal{N}\}$ so constructed satisfies the conditions in the lemma.

To get a T^{N_P}-invariant construction, we only need to run through above construction for $\mathcal{GM}(J, H; A)$ and define V_I as the lifting of the corresponding sets constructed by using stable $\mathcal{G}$-maps. $\Box$

Now we define

$$\widetilde{V}_I = (\pi_I)^{-1}(V_I), \quad \widetilde{E}_I = (\pi_I)^{-1}(\mathcal{L}_I).$$

Then the bundle $(\widetilde{E}_I, \widetilde{V}_I)$ are still a pair of (stratified) smooth varieties and for any $J \subset I$ the projection π^I_J still can be defined when restricted to $(\pi^I_J)^{-1}(\widetilde{E}_J, \widetilde{V}_J) \cap (\widetilde{E}_I, \widetilde{V}_I)$. Since locally $\widetilde{V}_I$ is a finite union of its (stratified smooth) components, we will say a continuous section $S_I : \widetilde{V}_I \to \widetilde{E}_I$ to be smooth if locally, S_I restricted to any of those components is (stratified)smooth. For a smooth section S_I, we say that S_I is transversal to zero section if locally, S_I restricted to any of the smooth components of $\widetilde{V}_I$ is transversal to zero section .

Now let $(\widetilde{E}, \widetilde{V})$ be the collection $\{(\widetilde{E}_I, \widetilde{V}_I), \pi^I_J; J \subset I \in \mathcal{N}\}$. We can define a global section $S = \{S_I; I \in \mathcal{N}\}$ of such a system by requiring the obvious compatibility condition:

$$(\pi^I_J)^* S_J = S_I|_{{\pi^I_J}^{-1}(\widetilde{V}_J)}.$$

S is said to be transversal to zero section if each S_I is.

Now the section $\overline{\partial}_{J,H} : W \to \mathcal{L}$ gives rise to a global section of the bundle system $(\widetilde{E}, \widetilde{V})$ in an obvious way. Our goal now is to perturb $\overline{\partial}_{J,H}$ to get a global transversal section. To this end, we need to know how an element $\nu_i \in K_i$ can be interpreted as a global section of $(\widetilde{E}, \widetilde{V})$ first.

Lemma 6.3 *Each $\tilde{\nu}_i \in \widetilde{K}_i$ gives rise to a global section, denoted by same notation $\tilde{\nu}_i = \{(\tilde{\nu}_i)_I; I \in \mathcal{N}\}$, of the system $(\widetilde{E}, \widetilde{V})$, which is T^{N_P}-equivariant.*

Proof:

By multiplying with some Γ_i-equivariant cut-off function β_i, we may assume that the support of each element $\tilde{\nu}_i$ is contained in $\widetilde{W}^1_i = \pi_i^{-1}(W^1_i)$ and that $\{U^0_i; i = 1, \cdots, q\}$ already forms a covering of $\mathcal{FM}(J, H, A)$, where $\widetilde{U}^0_i = \{u \mid u \in \widetilde{W}_i, \beta_i(u) > 0\}$ and $U^0_i = \pi_i(\widetilde{U}^0_i)$. Now since each ν_i vanishes near the boundary of $\widetilde{W}_i$, we may consider it as a global multi-valued section $\bar{\nu}_i$ of $\mathcal{L} \to W$ supported in $U^0_i \subset\subset W^1_i$. Let $I \in \mathcal{N}$ with $i \notin I$ and consider V_I.

Recall that if $I = \{i_1, \cdots, i_n\}$ then

$$V_I = W_{i_1}^n \cap W_{i_2}^n \cdots \cap W_{i_n}^n \setminus \cup_{J \in \mathcal{N}_{n+1}} Cl(U_{j_1}^n) \cdots \cap Cl(U_{j_{n+1}}^n)$$

with $J = (j_1, \cdots, j_{n+1})$. Since $i \notin I$,

$$\begin{aligned} V_I &\subseteq W_{i_1}^n \cap \cdots \cap W_{i_n}^n \setminus W_{i_1}^n \cap \cdots \cap W_{i_n}^n \cap Cl(W_i^1) \\ &\subseteq W \setminus Cl(W_i^1). \end{aligned}$$

Therefore, the intersection $Cl(U_i^0) \cap Cl(V_I) = \emptyset$. Hence $\tilde{\nu}_i|_{V_I} \equiv 0$ for any $I \in \mathcal{N}$ with $i \notin I$. We define $(\tilde{\nu}_i)_I \equiv 0$ if $i \notin I$.

Now assume that $i \in I$.

When $l(I) = 1$ hence $I = \{i\}$, $\widetilde{V}_I = \widetilde{V}_i$ and $(\tilde{\nu}_i)_I$ is just $\tilde{\nu}_i : \widetilde{W}_i \to \widetilde{\mathcal{L}}_i$ restricted to $\widetilde{V}_i$.

If we denote $\{i\}$ by I_i, then for any I with $n = l(I) > 1$, we have

$$\pi_{I_i}^I : (\widetilde{\mathcal{L}}_I^{\Gamma_I}, \widetilde{W}_I^{\Gamma_I}) \to (\widetilde{\mathcal{L}}_{I_i}, \widetilde{W}_{I_i}) \subset (\widetilde{\mathcal{L}}_i, \widetilde{W}_i).$$

Therefore, $(\pi_{I_i}^I)^*(\tilde{\nu}_i)_{I_i}$ gives rise to a section of $\widetilde{E}_I \to \widetilde{V}_I$, denoted by $(\tilde{\nu}_i)_I$. Clearly the section $(\tilde{\nu}_i)_I$, $I \in \mathcal{N}$ so constructed are compatible to each other and yields a well-defined global section $\tilde{\nu}_i = \{(\tilde{\nu}_i)_I, I \in \mathcal{N}\}$ of the system $(\widetilde{E}, \widetilde{V})$.

Finally, we note that β_i can be chosen to be T^{N_P}-equivariant, which implies that $\tilde{\nu}_i$ so constructed is also T^{N_P}-equivariant.

□

Let $K = \oplus_{i=1}^q K_i$. Consider the system

$$(\widetilde{E} \times \widetilde{K}_\delta, \widetilde{V} \times \widetilde{K}_\delta) = \{(\widetilde{E}_I \times \widetilde{K}_\delta, \widetilde{V}_I \times \widetilde{K}_\delta); I \in \mathcal{N}\}$$

of bundles, where $\widetilde{K}_\delta$ is a δ-neighborhood of zero of $\widetilde{K}$ under the identification of $\widetilde{K}$ and K. We now defined global section $\overline{\partial}_{J,H}^{\tilde{K}}$ given by

$$(\overline{\partial}_{J,H}^{\tilde{K}})(u_I, \tilde{\nu}) = \overline{\partial}_{J,H} u_I + \tilde{\nu}_I(u_I)$$

for any $(u_I, \tilde{\nu}) \in \widetilde{V}_I \times \widetilde{K}_\delta$.

Theorem 6.1 *$\overline{\partial}_{J,H}^{\tilde{K}}$ is a smooth section of $(\widetilde{E} \times \widetilde{K}_\delta, \widetilde{V} \times \widetilde{K}_\delta)$, which is transversal to zero section. It follows that when δ is small enough for a generic choice of the perturbation term $\tilde{\nu} \in \widetilde{K}_\delta$ the section $\overline{\partial}_{J,H}^{\nu} : \widetilde{V} \to \widetilde{E}$ is transversal to zero section and that the family of perturbed moduli spaces*

$$\widetilde{\mathcal{M}}^\nu = \{\widetilde{\mathcal{M}}_I^\nu = (\overline{\partial}_{J,H}^{\nu_I})^{-1}(0); I \in \mathcal{N}\}$$

is compatible in the sense that

$$\pi_J^I(\widetilde{\mathcal{M}}_I^\nu) = \widetilde{\mathcal{M}}_J^\nu \cap (Im \pi_J^I), \quad J \subset I.$$

Proof:

$\overline{\partial}^{\tilde{K}}_{J,H}$ is obviously (stratified) smooth. Since

$$(\overline{\partial}^{\tilde{K}_i}_{J,H})|_{\widetilde{U}_i^0} : \widetilde{U}_i^0 \times (\widetilde{K}_i)_\delta \to \pi_1^*(\widetilde{\mathcal{L}}_i|_{\widetilde{U}_i^0})$$

is transversal to zero section, so is $\overline{\partial}^{\tilde{K}}_{J,H}$ on $\widetilde{U}_i^0 \times \widetilde{K}_\delta$, for any $1 \le i \le q$.

Now $\{U_i^0; i = 1, \cdots, m\}$ already forms a covering of $\mathcal{FM}(J, H; A)$. Outside the inverse images of π_i of this covering, $\overline{\partial}^{\tilde{K}}_{J,H} = \overline{\partial}_{J,H}$. This implies that the zero set of $\overline{\partial}^{K}_{J,H}$ is contained in the inverse image of the covering. However, $\overline{\partial}^{K}_{J,H}$ is already transversal to zero section in $(\pi^I_{I_i})^{-1}(\widetilde{U}_i^0) \hookrightarrow \widetilde{W}_I$ for any $I \in \mathcal{N}$, where $I_i = \{i\}$. This proves the transversality for $\overline{\partial}^{\tilde{K}}_{J,H}$. It follows from implicit function theorem applied locally to each smooth component of $\widetilde{V} = \{\widetilde{V}_I; I \in \mathcal{N}\}$ that

$$(\overline{\partial}^{\tilde{K}}_{J,H})^{-1}(0) = \{(\overline{\partial}^{\tilde{K}}_{J,H})^{-1}_I(0); I \in \mathcal{N}\} \subset \widetilde{V} \times \widetilde{K}_\delta$$

is a family of "cornered" (stratified) smooth subvarieties.

Let

$$\pi : (\overline{\partial}^{\tilde{K}}_{J,H})^{-1}(0) \to \widetilde{K}_\delta$$

be the restriction of projection of $\widetilde{V} \times \widetilde{K}$ to $\widetilde{K}$. It is easy to see that Smale-Sard theorem is still applicable in this case. We conclude that for " generic" choice of $\tilde{\nu} \in \tilde{K}$, $\overline{\partial}^{\nu}_{J,H}$ is a transversal section of $(\widetilde{E}, \widetilde{V})$.

The compatibility of the family of zero set

$$\{\widetilde{\mathcal{M}}^{\nu}_I; I \in \mathcal{N}\} = \{\overline{\partial}^{\nu}_{J,H})^{-1}_I(0)\}$$

follows from the fact that $\overline{\partial}^{\nu}_{J,H}$ is a global section of $(\widetilde{E}, \widetilde{V})$.

□

We can give a canonical orientation for each $\tilde{\mathcal{M}}^{\nu_I}_I$ (see [F1], [FH]). Now we get a family of "singular cells" of $\mathcal{FB}(A)$, given by $\pi_I : \tilde{\mathcal{M}}^{\nu}_I \to \mathcal{FB}(A)$ for any $I \in \mathcal{N}$. If $I \subset J$, then π_I and π_J are related by the $\frac{|\Gamma_J|}{|\Gamma_I|}$-folded covering π^J_I in the overlap

$$(\pi^J_I)^{-1}(\tilde{\mathcal{M}}^{\nu}_I) \hookrightarrow \tilde{\mathcal{M}}^{\nu}_J.$$

This suggests that the family of the rational "singular" chains of $\mathcal{FB}(A)$, defined by $S^{\nu}_I = \frac{1}{|\Gamma_I|}\pi_I$ are compatible each other when restricted to those overlaps above. Therefore after been identified over those overlaps, $\{S^{\nu}_I,\ I \in \mathcal{N}\}$ form a well-defined rational relative "singular" cycle of $\mathcal{FB}(A)$.

We formally write this as

$$C^{\nu} = \sum_{I \in \mathcal{N}} S^{\nu}_I.$$

Now all T^{N_P}-actions carry over to the cycle C^ν. In particular there is an S^1-action on the top strata of C^ν. To state our main theorem, we need to indicate the dependence on the critical points c_+, c_- and homology class A in our notations. We will write $\tilde{\mathcal{M}}_I^\nu(c_-, c_+; A)$ and $S_I^\nu(c_-, c_+; A)$ etc. To define the boundary of $C^\nu(c_-, c_+; A)$, we also need to indicate where the fixed marking $x = (x_1, \cdots, x_n)$ is located in our notations. We have

Theorem 6.2 *For generic choice of (J, H, ν), fix any two critical points c_-, c_+ of H. There is a rational virtual moduli (relative) cycle*

$$C^\nu(c_-, c_+; A) = \sum_{I \in \mathcal{N}} S_I^\nu(c_-, c_+; A)$$

of $\mathcal{FB}(A)$, whose dimension is equal to $Ind(c_+) - Ind(c_-) + 2c_1(A) + 1$. The boundary $\partial C^\nu(c_-, c_+; A)$ is

$$\sum_{c, A_-, A_+} (C^\nu(c_-, c; A_-, x) \times C^\nu(c, c_+; A_+) \cup C^\nu(c_-, c; A_-) \times C^\nu(c, c_+; A_+, x),$$

$A_- + A_+ = A$, whose dimension is equal to $Ind(c_+) - Ind(c_-) + 2c_1(A)$. Moreover there are S^1 and T^2-actions on $C^\nu(c_-, c_+; A)$ and its boundary respectively.

We note that for any $\phi \in S^1$ and $g \in C^\nu(c_-, c_+; A)$, the action ϕ on g is given by a rotation of the domain $(\Sigma_g, l, x) \cong (S^1 \times \mathbf{R}; \tilde{l}, \tilde{x})$, which changes the relative position of the fixed marking x with respect to the marked line l. This implies that the S^1-action on $C^\nu(c_-, c_+; A)$ is free.

7 GW-invariants and Weinstein Conjecture

In this section we will prove Theorem 1 for genus zero case.

Using the moduli cycle we obtained in last section, we define a Morse theoretic version of GW-invariants under our main assumption that $\tilde{H}$ has no closed orbits.

Let $E_{\mathcal{F}} : \mathcal{FB}(A) \to V^n$. $F_l : \mathcal{FB}(A) \to \mathcal{GB}(A)$ given by forgetting the marked We now compose the moduli cycle $C^\nu(c_-, c_+; A, x)$ in $\mathcal{FB}(c_-, c_+; A)$ with the evaluation map $E_{\mathcal{F}}$ and define a (relative) moduli cycle $E_{\mathcal{F}} \circ C^\nu(c_-, c_+; A)$ in V^n. We denote it by $C_x^\nu = C_x^\nu(c_-, c_+; A)$.

Given $\beta_i \in H_*(V; \mathbf{Q}), i = 1, \cdots, n$, for simplicity, we may assume that each β_i can be represented by a smooth manifold of V. We still use β_i to denote this representative. Then $\beta = \beta_1 \times \cdots \beta_n$ is a cycle in V^n. Assume that the codimension of β in V^n is

$$\dim C_x^\nu(c_-, c_+; A) - 1. \qquad (**)$$

Now consider C_x^ν as a map from the " domain" $\frac{1}{|\Gamma|}\tilde{\mathcal{M}}^\nu = \sum_{I \in \mathcal{N}} \frac{1}{|\Gamma_I|}\tilde{\mathcal{M}}_I^\nu$ to V^n. By perturbing β_i slightly, we may assume that C_x^ν is transversal to β and

$(C_x^\nu)^{-1}(\beta)$ is a compact submanifold of $\frac{1}{|\Gamma|}\tilde{\mathcal{M}}^\nu$. The dimension assumption above implies that $\dim(C_x^\nu)^{-1}(\beta) = 1$. Since C_x^ν is S^1-equivariant, $(C_x^\nu)^{-1}(\beta)$ also carried an S^1-action, which is free as we mentioned above. This implies that $(C_x^\nu)^{-1}(\beta)/S^1$ is a finite set. We will use $C^\nu(c_-, c_+; A, \beta)$ to denote this set. There is an induced orientation on $C^\nu(c_-, c_+; A, \beta)$.

Definition 7.1 *Let (J, H) be a generic pair with H being C^∞-close to $\tilde{H}$. Given a homology class $A \in H_2(V; \mathbf{Z})$ and a cycle and a cocycle of Morse-Witten complex, represented by certain linear combinations $c-_, c_+$ of critical points of H, choose $\beta_i \in H_*(V; \mathbf{Q}), i = 1, \cdots, n$, such that (**) above holds. We define the Morse theoretical version of GW-invariant $\Phi_{A,J,H}(c_-, c_+)$ by specify its value at all such β's.*

$$\Phi_{A,J,H}(c_-, c_+, \beta_1, \cdots, \beta_n) \equiv \#(C^\nu(c_-, c_+; J, H, A, \beta)) \in \mathbf{Q},$$

where $\nu \in K_\epsilon$ is a generic element of K_ϵ. Note that on the righthand side of above equality, the number is counted with sign.

Now let $H_\lambda = \lambda \cdot H$, and choose a corresponding generic J_λ. We get a λ-dependent generic pair (J_λ, H_λ). Note that all $H_\lambda, \lambda > 0$, has same critical point set. We can define a λ-dependent GW-invariant $\Phi_{A,J_\lambda,H_\lambda}(c_-, c_+)$.

The key point needed to prove Theorem 1 is the following invariant property of the GW-invariants.

Theorem 7.1

$$\Phi_{A,J_\lambda,H_\lambda}(c_-, c_+, \beta_1, \cdots, \beta_n)$$

is independent of the choice of $\lambda, \lambda \in [\epsilon, \lambda_0 + 1/2)$, when (J_λ, H_λ) is generic.

Proof:

Fix $0 < \lambda_- < \lambda_+$, let $\Lambda = [\lambda_-, \lambda_+] \subset [\epsilon, \lambda_0 + 1/2)$ be the interval where the parameter λ varies. We can run through everything developed in the previous sections to incorporate the parameter λ. Therefore, we will have $\mathcal{FB}_\Lambda(A)$, $\mathcal{FM}_\Lambda(c_-, c_+, J_\lambda, H_\lambda, A)$ etc. Here for instance, $\mathcal{FB}_\Lambda(A) = \{(f, \lambda) \,|\, f \in \mathcal{FB}(A), \lambda \in \Lambda\}$. We can similarly define the virtual (relative) moduli cycle $C_\Lambda^\nu(c_-, c_+, J_\lambda, H_\lambda; A)$, $C_{x,\Lambda}^\nu(c_-, c_+; J_\lambda, H_\lambda, A)$ and $C_\Lambda^\nu(c_-, c_+; J_\lambda, H_\lambda, A, \beta)$. We may assume that ν has been chosen is such a way that at two end points λ_- and λ_+ of Λ, ν_{λ_-} and ν_{λ_+} are also generic so that $C^{\nu_{\lambda_-}}(c_-, c_+, J_{\lambda_-}, H_{\lambda_-}, A, \beta)$ and $C^{\nu_{\lambda_-}}(c_-, c_+, J_{\lambda_+}, H_{\lambda_+}, A, \beta)$ are well-defined. Now the crucial step is the following

Lemma 7.1 *When the condition (**) on dimension holds, $C_\Lambda^\nu(c_-, c_+; J_\lambda, H_\lambda, A, \beta)$ is a one dimensional (relative) virtual moduli cycle. It has the boundary*

$$\begin{aligned}
& \partial C_\Lambda^\nu(c_-, c_+; J_\lambda, H_\lambda, A, \beta) \\
= \; & C^{\nu_{\lambda_-}}(c_-, c_+; J_{\lambda_-}, H_{\lambda_-}, A, \beta) \\
\cup \; & C^{\nu_{\lambda_+}}(c_-, c_+, J_{\lambda_+}, H_{\lambda_+}, A, \beta) \\
\cup \; & \{\cup_{\lambda,c} M(c_-, c; H_\lambda) \times C^{\nu_\lambda}(c, c_+; J_\lambda, H_\lambda, A, \beta)\} \\
\cup \; & \{\cup_{\lambda,c} C^{\nu_\lambda}(c_-, c; J_\lambda, H_\lambda, A, \beta) \times M(c, c_+; H_\lambda)\},
\end{aligned}$$

where in the third term c runs through all critical points of H_λ such that $Ind(c) - Ind(c_-) = 1$ and in the fourth term $Ind(c_+) - Ind(c) = 1$. Here $M(c_-, c; H_\lambda)$ is the moduli space of unparametrized gradient lines of ∇H_λ connecting c_- and c. So is $M(c, c_+; H_\lambda)$ in a similar way.

Proof:

The boundary $\partial C^\nu_\Lambda(c_-, c_+; J_\lambda, H_\lambda, A, \beta)$ certainly contains the four terms listed in the lemma. We need to prove that there is no other terms. Consider one of the components in the boundary $\partial C^\nu_\Lambda(c_-, c_+; J_\lambda, H_\lambda, A, \beta)$. By an analogy of Theorem 6.2. we may assume that it has the form

$$\cup_{\lambda_i} C^{\nu_{\lambda_i}}(c_-, c; A_-) \times C^{\nu_{\lambda_i}}(c, c_+; A_+, x),$$

where $A_- + A_+ = A$ and $\lambda_i \in \Lambda$ are of finitely many. Now the evaluation map

$$E = \cup_{\lambda_i} E_{\lambda_i} : \cup_{\lambda_i} C^{\nu_{\lambda_i}}(c_-, c; A_-) \times C^{\nu_{\lambda_i}}(c, c_+; A_+, x) \to V^n$$

only involves the second components above. The dimension condition (**) implies that after quotienting out the S^1-action of second factor, $E^{-1}(\beta)$ is already zero dimensional and hence is a finite set for our generic choice (J, H, ν). Now for any element $g \in E^{-1}(\beta)/S^1$, there is another S^1-action acting on the first factor. However the isotropy group $I_g \hookrightarrow S^1$ is either finite or S^1. The first case contradicts to the finiteness of $E^{-1}(\beta)/S^1$. This implies that any element g in first factor $C^{\nu_i}(c_-, c, A_-)$ is S^1-invariant. This, in turn, implies that $A_- = 0$, $A_+ = A$, and $Ind(c) - Ind(c_-) = 1$. Now $M(c_-, c; H_{\lambda_i})$ is obtained in $\mathcal{FM}(c_-, c; J_\lambda, H_\lambda, 0)$ as an isolated compact component when $Ind(c) - Ind(c_-) = 1$, and it is just the fixed points set of the S^1-action. It follows from this and the vanishing property along $M(c_-, c; H_{\lambda_i})$ of elements in K that

$$C^{\nu_{\lambda_i}}(c_-, c_+; A_-) = M(c_-, c; H_{\lambda_i}).$$

Note that in the last two terms of the expression for the boundary

$$\partial C^\nu_\Lambda(c_-, c_+; J_\lambda, H_\lambda, A_\lambda),$$

there are only finite many $\lambda \in \Lambda$ involved for the dimensional reason.

□

We now prove that counting algebraically, the last two terms have no contribution to the boundary operator. Recall that we have defined c_- and c_+ as two cycles of the Morse-Witten complex with respect to Morse function H and $-H$ respectively. This implies that when $Ind(c) - Ind(c_-) = 1$ or $Ind(c_+) - Ind(c) = 1$, $\#(M(c_-, c; H)) = 0$, $\#(M(c, c_+; H)) = 0$, where both numbers are counted algebraically. Therefore for any fixed λ_i and fixed critical point c_-, c of H_{λ_i} with $Ind(c) - Ind(c_-) = 1$,

$$\#\{M(c_-, c; H_{\lambda_i}) \times C^{\nu_{\lambda_i}}(c, c_+; J_{\lambda_i}, H_{\lambda_i}, A, \beta)\} =$$
$$\#\{M(c_-, c; H_{\lambda_i})\} \times \#\{C^{\nu_{\lambda_i}}(c, c_+; J_{\lambda_i}, H_{\lambda_i}; A, \beta)\} = 0.$$

This proves that the boundary operator, in the sense of algebraic topology, gives

$$\begin{aligned}\partial C^{\nu}_{\Lambda}(c_-,c_+;J_\lambda,H_\lambda,A,\beta) &=\\ C^{\lambda_+}(c_-,c_+;J_{\lambda_+},H_{\lambda_+},A,\beta)&\\ -C^{\lambda_-}(c_-,c_+;J_{\lambda_-},H_{\lambda_-},A,\beta)&\end{aligned}$$

as rational virtual moduli cycles.

The invariance of $\Phi_{A,J_\lambda,H_\lambda}(c_-,c_+)$ about λ follows.

$\square$

Proof of Theorem 1.1 in the case of genus zero:

Recall that c_- and c_+ are the Morse theoretic representations of two homology classes α_- and α_+ with $supp(\alpha_-)\subset V_+$ and $supp(\alpha_+)\subset V_-$. We have assumed that the usual GW-invariant $\Psi_{A,n+2}(\alpha_-,\alpha_+)\neq 0$, hence $\Psi_{A,n+2}(\alpha_-,\alpha_+,\beta_1,\cdots,\beta_n)\neq 0$ for certain β_i's. The usual GW-invariant for general symplectic manifolds are established in the work of [FO] and [LiT]. We refer readers to these references for the relevant definition.

As we mentioned in Theorem 1.5 whose proof is in [LiuT3], that when $0<\lambda$ is small enough, we have

$$\Psi_A(\alpha_-,\alpha_+,\beta_1,\cdots,\beta_n)=\Phi_{A,J_\lambda,H_\lambda}(c_-,c_+,\beta_1,\cdots,\beta_n).$$

Theorem 7.1 and our assumption imply that $\Phi^{\lambda}_{A,J_\lambda,H_\lambda}\neq 0$ for any $\lambda\in\Lambda$. However, we will prove in a moment

Lemma 7.2 *When $\lambda\in\Lambda$ is large enough, the moduli space $\mathcal{FM}(c_-,c_+;J_\lambda,H_\lambda,A)$ is empty.*

This implies that

$$\Phi_{A,J_\lambda,H_\lambda}(c_-,c_+,\beta_1,\cdots,\beta_n)=0$$

when $\lambda\in\Lambda$ is large enough. We get a contradiction. This implies that our main assumption that S and hence $\tilde H$ has no closed orbits is not correct. This finishes the proof of Theorem 1.

To prove this last lemma, choose an element $f\in\mathcal{FM}(c_-,c_+;J_\lambda,H_\lambda,A)$, and calculate its energy. Assume that $f=\cup_{i=1}^{N_P}f_i^P\cup_{j=1}^{N_B}f_j^B$ with f_i^P connecting critical points c_i and c_{i+1} of $H_\lambda=\lambda\cdot H$. Since $\sum_i[f_i^P]+\sum_j[f_j^B]=A$, we have

$$\begin{aligned}0\le E(f) &= \sum_i E(f_i^P)+\sum_j E(f_j^B)\\ &= \sum_i(H_\lambda(c_{i+1})-H_\lambda(c_i)+\omega([f_i^P]))+\sum_j\omega([f_j^B])\\ &= \lambda(H(c_+)-H(c_-))+\omega(A). \qquad (Main\,Estimate)\end{aligned}$$

Therefore,

$$\lambda \leq \frac{\omega(A)}{H(c_-) - H(c_+)}.$$

Now H is C^0-closed to the given $\tilde{H}$, which implies that

$$\frac{\omega(A)}{H(c_-) - H(c_+)} \leq \frac{\omega}{\tilde{H}(c_-) - \tilde{H}(c_+)} + \frac{1}{2} = \frac{\omega(A)}{2\epsilon} + \frac{1}{2}.$$

Note that since $\Psi_A \neq 0$, the class A can be represented by some J- holomorphic sphere. We have $\omega(A) > 0$. We conclude that if $\mathcal{FM}(c_-, c_+; J_\lambda, H_\lambda, A)$ is not empty, then $0 < \lambda < \frac{\omega(A)}{2\epsilon} + \frac{1}{2}$. This proves the lemma.

□

8 Higher Genus Case

In this section, we will prove Theorem 1 in higher genus case. Our main observation here is that the main energy estimate can be carried out in a coordinate-free manner. In particular, existence of a preferable cylindrical coordinate of $S^2 \setminus \{-\infty, +\infty\}$ in genus zero case does not play any essential role in the estimate.

The method we developed in the previous sections can be adapted to higher genus to define the GW-invariants $\Psi_{A,g,n+2}$ (see [FO] and [LT] for the details and other methods).

To define higher genus perturbed GW-invariants $\Phi_{A,J,H,g,n+2}$, we need to modify the definition of (J, H)-maps of genus zero case. For that purpose, we need to find certain 1-forms on (Σ, j) as a replacement of ds and $d\theta$. Let $\bar{\mathcal{M}}_{g,n+2}$ be the Deligne-Mumford copmpactification of $\mathcal{M}_{g,n+2}$ of stable curves of genus g with $n + 2$ (ordered) marked points. We will use $-\infty$ and $+\infty$ to denote the first two marked points. It is well-known that $\bar{\mathcal{M}}_{g,n+2}$ is an orbifold. We can define the orbifold bundle of closed 1-forms with poles at double points, $\mathcal{CF}_{n+2} \to \bar{\mathcal{M}}_{g,n+2}$ as follows.

Given any $\langle\Sigma, j\rangle \in \bar{\mathcal{M}}_{g,n+2}$, let (Σ, j) be a representative of it. The fiber $(\mathcal{CF}_{n+2})_{\langle\Sigma,j\rangle}$

$$= \{\xi | \xi \text{ is a closed 1-form over } \Sigma \setminus \{\text{double points}, -\infty, +\infty\},\ Res(\xi, d) = 0\}$$

modulo the equivalence relation induced by the action of the automorphism group of (Σ, j), where d is a double point of Σ and the residue $Res(\xi, d)$ is defined to be $\int_{C_+} \xi + \int_{C_-} \xi$. Here C_+ and C_- are the boundaries of small discs $D_+$and D_- of Σ centered at d, oriented by the induced orientation of D_+ and D_-. Note that here we consider $-\infty$ and $+\infty$ as a single double point.

Given $(\Sigma, j) \in \langle\Sigma, j\rangle \in \bar{\mathcal{M}}_{g,n+2}$, let $\widetilde{W}_\epsilon(\Sigma)$ be a local uniformizer of $\bar{\mathcal{M}}_{g,n+2}$ near Σ with cover group Γ_Σ. For simplicity, we may assume that Σ has $3g -$

$3+n+2$ double points. Then the local deformation of Σ can be completely described by the gluing process as we did for genus zero case, with the gluing parameter t of $3g-3+n+2$ complex components. We will use Σ_t to denote the corresponding deformation. Let $D_{l,+}$ and $D_{l,-}$ be two fixed discs at a double point $d_l \in \Sigma$. After gluing, we get a corresponding annulus $C_{l,t} \cong S^1 \times [0, L_l(t)]$ in Σ_t. By taking the intersection of the Γ_Σ-orbit of $C_{l,t}$, we may assume that each $C_{l,t}$ is Γ_Σ-invariant. In general cylindrical coordinate (s_l, θ_l) along $C_{l,t}$ is only defined modulo the action of Γ_Σ. Since the action preserves marked points, the cylindrical coordinates near the two ends $-\infty$ and $+\infty$ are always well -defined for any elements in $\bar{\mathcal{M}}_{g,n+2}$.

Proposition 8.1 *There exists a continuous section Y^* of the bundle $\mathcal{CF} \to \bar{\mathcal{M}}_{g,n+2}$ such that the value of Y^* (lifted to an uniformizer) at any $\Sigma_t \in \widetilde{W}_\epsilon(\Sigma)$ has the property that*

$$Y^*(\Sigma_t)|_{C_{l,t}} = a_l d\theta_l,$$

$$Y^*(\Sigma_t)|_{C_{-\infty,t}} = d\theta_{-\infty},$$

$$Y^*(\Sigma_t)|_{C_{+\infty,t}} = d\theta_{+\infty},$$

where a_l is a constant.

Proof:

Step I. Construct Y^* on each $\widetilde{W}_\epsilon(\Sigma)$.

We cosider the case that $n=0$ first.

Fix a smooth $\Sigma_{t_0} \in \widetilde{W}_\epsilon(\Sigma)$, let $C_l = C_l(t_0)$, $l = 1, \cdots, L$, be the all simple closed geodesic of $(\Sigma, m(j_0))$, where $m(j_0)$ is the hyperbolic metric which corresponds to the complex structure on Σ_{t_0}. The degeneracy from Σ_{t_0} to Σ can be described by shrinking $3g-3+2$ many of C_l's of Σ_{t_0}. For simplicity, we assume that first $3g-3+2$ C_l's are to be shrunk to double points of Σ. We may assume that the length of $C_l(t)$ is bounded above for $\Sigma_t \in \widetilde{W}_\epsilon(\Sigma)$ and the upper bound has been achieved at Σ_{t_0}. Then each element $\Sigma' \in \widetilde{W}_\epsilon(\Sigma)$ can be obtained from Σ_{t_0} by cutting along some of $C_l, l = 1, \cdots, L$, and insert some neck $N_i \cong S^1 \times [0, M_i]$ through gluing along boundaries. Therefore we only need to construct Y^* at Σ_{t_0} with described property of theorem. To see this, we note that by using the local translation invariance of $a_l d\theta_l = Y^*(\Sigma_{t_0})|_{C_{l,t_0}}$ one can easily extend Y^* over N_i, hence define $Y^*(\Sigma')$ for any $\Sigma' \in \tilde{W}_\epsilon(\Sigma)$.

To construct $Y^*(\Sigma_{t_0})$, we cut off $D_{+\infty}, D_{-\infty}$ of the neighbourhoods of $+\infty$ and $-\infty$ and glue their boundary $C_{+\infty}$ and $C_{-\infty}$ back to get a curve $\tilde{\Sigma}$ of genus $g+1$ with $C_0 = C_{+\infty} = C_{-\infty} \hookrightarrow \tilde{\Sigma}$. By our assumption all $C_l \hookrightarrow \Sigma, 1 \le l \le L$, is still contained in $\tilde{\Sigma}$ and $C_0 \cap C_l = \emptyset$. Let $e_1, \cdots, e_{2g+2}$ be the generators of $H_1(\tilde{\Sigma})$. Write $C_l = \sum_j a_{lj} e_j, 0 \le l \le L$. Since each $[C_l] \neq 0$ in $H_1(\tilde{\Sigma}), \mathbf{Z})$, for any fixed l, there exist some j, $1 \le j \le 2g+2$ such that $a_{l,j} \neq 0$. Clearly, there exist some $x = (x_1, \cdots, x_{2g+2})$ such that $\sum_j a_{0,j} x_j = 1$ and $\sum_j a_{l,j} x_j \neq 0, 1 \le l \le L$. By de Rham theorem, we can find a $\tilde{Y}^*$ such that $\langle \tilde{Y}^*, e_j \rangle = x_j$. This implies that

$\langle \tilde{Y}^*, C_0 \rangle = 1$ and $\langle \tilde{Y}^*, C_l \rangle \neq 0$, $1 \leq l \leq L$. By adding an exact one-form and back to Σ_{t_0} , we can find a $\tilde{Y}^*(\Sigma_{t_0})$ with the desired property, hence a section $\tilde{Y}^*$ on $\tilde{W}_\epsilon(\Sigma)$. This completes the local construction for $n = 0$.

To deal with the general case, we may assume, for simplicity, that $\Sigma_{t_0} = \Sigma^1_{t_0} \cup \Sigma^2_{t_0}$, where $\Sigma^1_{t_0}$ is a genus g curve with $k+2$ marked points, $k < n$, without any unstable rational components after forgetting its last k marked points, and $\Sigma^2_{t_0}$ consists of only stable rational components (bubbles) carrying the rest $n-k$ marked points. We may assume further that $\Sigma^2_{t_0}$ has only one component with two marked points and one double point d. The local deformation of Σ_{t_0} consists of two parts: the deformation of $\Sigma^1_{t_0}$ in $\bar{\mathcal{M}}_{k+2}$, with the resulting surface $\Sigma^1_{t^1}$ associated to the complex gluing parameter t^1 of $3g - 3 + k + 2$ components, and a further deformation of $\Sigma^1_{t^1} \cup \Sigma^2_{t_0}$ with a gluing parameter t^2 associated to the double point d. It is easy to see that we can define Y^* for all local deformation $\Sigma^1_{t^1}$ by pulling back the Y^* defined for the case $n = 0$ through the local projetion given by forgetting the last k marked points. We define Y^* at $\Sigma^1_{t^1} \cup \Sigma^2_{t_0}$ by declaring its value to be zero along $\Sigma^2_{t_0}$. Extending Y^* over the final deformation given by t^2 can be easily obtained. We leave it to the reader.
Step II.

By taking the average of the action Γ_Σ on $\tilde{Y}^*$, we get a section Y^* over $W_\epsilon(\Sigma) = \tilde{W}_\epsilon(\Sigma)/\Gamma_\Sigma$, with the desired property.

By using a partition of identity subject to a finite covering of $\bar{\mathcal{M}}_{g,2}$ given by $\{\tilde{W}_{\epsilon_i}(\Sigma_i)/\Gamma_{\Sigma_i}\}$, we can paste these local Y^* defined on $\tilde{W}_{\epsilon_i}(\Sigma_i)$ together to get a well-defined Y^* with the required property.

$\square$

What we need is slightly more. We need to define $Y^*(\Sigma)$ for any semi-stable curve Σ of genus g. Each semi-stable curve Σ can be obtained from some stable curve Σ' by inserting first some unstable principal components $(P_l; (z_l)_-, (z_l)_+)$'s with $P_l \setminus \{(z_l)_-, (z_l)_+\} \cong \mathbf{R}^1 \times S^1$ at a double point or at the two ends of Σ', then adding some bubble components B_j's. Clearly, Y^* can be extended in an obvious way to include all semi-stable curves in its domain.

We will use Σ_l to denote component Σ' and write $\Sigma = \cup_l \Sigma_l \cup_i P_i \cup_j B_j$. We now define (j, J, H)-map f with domain (Σ, j) of semi-stable curve of genus g by using the following equations:
(i) on B_j, $df^B_j + J(f^B_j) \circ df \circ j = 0$;
(ii) on Σ_l, $df^\Sigma_l + J(f^\Sigma_l) \circ df \circ j - \nabla H(f^\Sigma_l) j^*(Y^*) + J(f^\Sigma_l) \nabla H(f^\Sigma_l) Y^* = 0$;
(iii) on P_i, same as (ii) for f^P_i.

As in genus zero case, we impose the obvious asymptotic condition along all ends of Σ.

By using local convergence described in Section 1, together with the local translation invariance of Y^* along necks, we can prove the Gromov-Floer compactness theorem for (j, J, H)-maps. We remark that each (j, J, H)-map f is defined on $\Sigma \setminus \{\text{double points}\} \cup \{-\infty, +\infty\}$ and we consider double points of Σ and $-\infty, +\infty$ as ends of f. Along those ends f^Σ_l or f^P_i are convergent to

(successive) critical points of ∇H, if the corresponding $a_l \neq 0$. Otherwise, they are (j, J)-holomorphic along these ends, hence can be extended smoothly over the double points.

Set $X^* = j^* Y^*$. Let $X, Y = j(X)$ be the dual vector fields of X^* and Y^*. We define energy

$$\begin{aligned} E(f) &= \sum_i \int\int_{P_i} \langle df_i^P(X), df_i^P(X)\rangle X^* \wedge Y^* \\ &+ \sum \int\int_{\Sigma_l} \langle df_l^\Sigma(X), df_l^\Sigma(X)\rangle X^* \wedge Y^* + \sum_j \int\int_{B_j} (f_j^B)^* \omega \end{aligned}$$

Since $X^* \wedge Y^*$ is compatible with the orientation of P_i or Σ_l at its non-zero points, we have $E(f) \geq 0$.

We can now recover our main estimate.

Theorem 8.1 *For $\lambda > \lambda_0 + \frac{1}{2}$, there is no $(j, J_\lambda, H_\lambda)$-map connecting c_- and c_+ of class A.*

Proof:

$$\begin{aligned} 0 \leq E(f) &= \int\int_B f^*\omega + \int\int_{P \cup_l \Sigma_l} \omega(df(X), df(Y)) X^* \wedge Y^* \\ &+ \lambda \int\int_{P \cup_l \Sigma_l} \langle \nabla H, df(X)\rangle X^* \wedge Y^* \\ &= \int\int_\Sigma f^*\omega + \lambda \int\int_{P \cup_l \Sigma_l} (d(H \circ f)(X) X^* + d(H \circ f)(Y^*)) \wedge Y^* \\ &= \int\int_\Sigma f^*\omega + \lambda \int\int_{P \cup_l \Sigma_l} d(H \circ f \cdot Y^*) \\ &= \omega(A) + \lambda(H \circ f(+\infty) - H \circ f(-\infty)) \\ &+ \lambda \sum_i (H \circ f(d_i) Res(Y^*(\Sigma), f(d_i))) \\ &= \omega(A) + \lambda(H(c_+) - H(c_-)). \end{aligned}$$

This implies that

$$\lambda \leq \frac{\omega(A)}{H(c_-) - H(c_+)} \leq \lambda_0.$$

□

By using $(j, J_\lambda, H_\lambda)$-maps, we can now repeat our construction from Section 3 to Section 7 to get a virtual moduli cycle in higher genus case and to define perturbed GW-invariant $\Phi_{A, J_\lambda, H_\lambda, g, n+2}$ for $\epsilon \leq \lambda \leq \lambda_0 + \frac{1}{2}$. What is left is to prove the higher genus case of Theorem 1.6.

Theorem 8.2

$$\Phi_{A,J_\lambda,H_\lambda,g,n+2}(c_-,c_+,\beta_1,\cdots,\beta_n)$$

is well-defined, independent of the choices of $\lambda\in[\epsilon,\lambda_0+\frac{1}{2}]$*. It is equal to zero when* $\lambda>\lambda_0$*.*

Proof:

The last statement follows from our estimate above. The first part can be proved in a similar way as before. It fact, if there is no unstable principal component f_i^P appearing as connecting orbits, th invariance of $\Phi_{A,J_\lambda,H_\lambda,g,n+2}$ with respect to λ can be proved in same way as the usual GW-invariant $\Psi_{A,J_\lambda,g,n+2}$. For each fixed intersection pattern, those unstable principal components f_i^P of connecting orbits form several strings, each connecting two critical points of H. We may apply the theory for genus zero case to each of those strings separately. The general case now follows by combining above two cases.

□

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